THE DIFFRACTION OF LIGHT BY ULTRASONIC WAVES

Thesis

Submitted by

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PREFACE.

In this thesis, the problem of the diffraction of light by ultrasonic waves has been rigorously formulated in terms of the scattering of electromagnetic waves by a periodically perturbed medium. It is shown that the present theory is valid and useful over the entire range of experimental conditions where the phenomenon of ultrasonic diffraction is found to occur; this is in contrast to the existing theories which are each at best valid over different parts of this range. Various conclusions of the theory are found to be in satisfactory agreement with the experimental results.

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I. INTRODUCTION

Through the study of the scattering of light by thermal vibrations in liquids, Brillouin (1921) was led to postulate the diffraction of light by ultrasonic waves many years before its experimental realization by Debye and Sears (1932) and Lucas and Biquard (1932). The experimental work showed, at once, an unexpected variety and complexity of spectra.

In the first theoretical treatment of this problem, Brillouin (1933) set up Maxwell's equations for the region disturbed by the ultrasonic beam and considered the propagation of an incident plane wave in the disturbed medium. The dielectric constant occurring in the wave equations was assumed to be directly proportional to the density of the medium. Brillouin solved these equations by two approximate methods. In the first method he assumed (a) that the incident wave kept its constant amplitude inside the medium independent of the thickness of the ultrasonic beam, and (b) that the density variations in the ultrasonic beam were small enough to justify the consideration of the emission of secondary waves by the incident wave alone, that is, the amplitude of the secondary waves was assumed to be considerably smaller than that of the incident wave. This approximate method gives only first order lines. In the same paper, Brillouin realized that this approximation was rather crude; he, therefore, disregarded the assumptions (a) and (b) above and obtained the intensities of various orders in terms of power series of the parameter,

$$\delta = \Delta \Lambda^2 / \lambda^2 \tag{1}$$

where Δ is the ratio of the maximum density change to the density of the undisturbed medium, Λ is the wave-length of the sound and λ the vacuum wave-length of the incident light. Such expressions in terms of power series have a limited range of usefulness since they converge rapidly only if $\delta \ll 1$.

The second theoretical treatment came from Raman and Nath (1936-38); they started from the scalar wave equation and assumed that the refractive index of the medium was directly proportional to its density. They succeeded in reducing this equation to a set of difference differential equations

$$\left.\begin{aligned} &2\phi_\ell' - (\phi_{\ell-1} - \phi_{\ell+1}) = c_\ell \phi_\ell \\ &\text{where } c_\ell = \frac{i}{n^2 \delta}\left(\ell^2 + 2\ell n \frac{\Lambda}{\lambda}\theta\right). \end{aligned}\right\} \tag{2}$$

Here ϕ_ℓ is the amplitude of the wave of ℓ th order, θ is the angle of incidence, n the refractive index of the unperturbed medium, and ϕ' denotes differential coefficient of ϕ with respect to its argument $2\pi\Delta Z/\lambda$ and Z is the distance traversed by light in the ultrasonic region. Raman and Nath then neglected the term $c_\ell\phi_\ell$ occurring on the right hand side of (2) and thus obtained for ϕ_ℓ closed expressions in terms of Bessel functions of integral order. This procedure does not appear to be correct since, even if $1/\delta$ be small compared to unity, ℓ^2/δ will not be negligible for sufficiently large ℓ. For example, if $1/\delta \sim 10^{-2}$ or greater, the Raman and Nath expressions for ϕ_ℓ have no adequate theoretical justification. However for $1/\delta \ll 1$ and small values of ℓ, their Bessel function solutions for ϕ_ℓ might form a good approximation.

It may also be mentioned that Raman and Nath (1935), in an earlier treatment of the problem, based on work by Rayleigh (1907) on phase gratings, obtained the diffraction wave amplitudes in terms of Bessel functions of the same argument as above. This method considered only the phase changes as a plane wave traversed the ultrasonic beam.

It will be noticed that the Raman and Nath approximation, $\delta \gg 1$, and the Brillouin approximation, $\delta \ll 1$, could, at best, be valid and useful only under opposite conditions. In the intermediate range of values of the parameter δ, ($\delta \sim 1$) neither treatment is able to give expressions for the intensities of the different orders. Moreover, they give no indication of the number of orders likely to appear for a given value of the parameter δ, i.e., under given experimental conditions (except, of course, in the trivial case of $\delta \ll 1$).

Various authors (notably Extermann and Wannier (1936), Nath (1936), David (1937), van Cittert (1937), Rytov (1938) and Mertens (1951)) have tried to improve upon these approximations. None of them succeeded, however, in obtaining convenient expressions for the calculation of the intensities of the various order lines over the entire range of the parameter δ.

In the present work, we approach the problem of the diffraction of light by ultrasonic waves, in a different manner, suggested by a paper of Darwin (1924) on the optical constants of matter. Darwin considers the problem of reflection and refraction of light through a homogeneous medium in the following way: When a light wave is incident on the medium, each element of volume of the medium may be regarded as emitting secondary waves (the amplitude of these secondary waves being determined by the scattering index τ) and each element of volume is under the exciting force of not only the incident wave, but also of all the secondary waves emitted by the other elements of volume of the medium. This leads, as is frequently the case in rigorous diffraction theory (See e.g., Clemmow, 1951), to an integral equation for the unknown electromagnetic disturbance $\underline{E}$ (or $\underline{H}$) in the medium. After $\underline{E}$ has been determined from the integral equation, one can calculate the intensities of reflected and refracted waves by suitably combining the secondary waves outside the medium due to $\underline{E}$. For a <u>homogeneous</u> medium, considered by Darwin, the scattering index τ turns out to be given by

$$\tau = \frac{3\,(n^2-1)}{(n^2+2)} = \tau_0 \quad \text{say} \tag{3}$$

Now we know from the Lorentz-Lorenz law that $(n^2-1)\,/\,(n^2+2)$ is directly proportional to the density of the medium. It is, therefore, physically reasonable to assume that, for a heterogeneous medium, the scattering index at any point is directly proportional to the density at that point. No such physical argument can be advanced for the corresponding assumptions, of earlier workers quoted above, relating to the proportionality of refractive index or dielectric constant to the density of the perturbed medium.

In this thesis, we start from the proportionality of the scattering index to the density and obtain the integral equation for our problem. As a trial solution for this equation, we take the resultant electromagnetic disturbance E in the medium as a superposition of a discrete set of plane waves. The condition for the self-consistency of the trial solution gives, (a) the possible frequencies and directions of the diffracted waves and (b) an infinite set of equations for the amplitudes of the plane waves forming the trial solution. It is shown that these equations are sufficient to determine uniquely these amplitudes and, hence, the amplitudes of the diffracted waves. Although it has not been possible to solve the infinite set of equations for the amplitudes explicitly and obtain analytical expressions for them, it is shown that, for a given value of the parameter δ and for given angle of incidence θ, a finite number of these equations suitably chosen from the set is sufficient to give a good approximation for the amplitudes. This gives also an indication as to the number of orders likely to appear under given experimental conditions, i.e. for given values of δ and θ. In particular, the well-known asymmetry in the different spectra for oblique incidence receives a natural explanation. The estimated asymmetry as a function of the angle of incidence is found to be in qualitative agreement with the experimental results.

On the other hand, explicit expressions for the intensities of the first order lines for both normal and Bragg incidence can be easily calculated on the basis of our theory; this calculation is given in §3.3. These, as well as various other theoretical results, are compared with the experimental facts and are found to be in satisfactory agreement.

Finally, we mention that we treat only the case of the E-polarization. The discussion of the H-polarization is analogous, though somewhat more complicated, and will not be given.

II. A DYNAMICAL THEORY OF ULTRASONIC DIFFRACTION.

We treat the diffraction of light by ultrasonic waves in liquids as a classical scattering of electromagnetic waves by the elementary scattering centres or "particles" of the medium. In this method the direct combination of the waves scattered by the particles requires no introduction of boundary conditions at refracting surfaces (disturbed regions at the edge of the ultrasonic beam) but brings in the ultrasonic beam thickness through the process of integration over the scattering volume. Moreover, one makes no appeal to any specific scattering mechanism such as is given by the classical theory of electrons or the quantum theory of radiation, but introduces a scattering index τ appropriate for the medium. We now proceed to define τ .

§2.1. The Scattering Index τ.

The scattering index τ is defined in the following manner. Let an element of volume dV at (x,y,z) be illuminated by light with components $E_\alpha\,(t,x,y,z)$. Then the scattered light at the point (x',y',z'), a distance r from the infinitesimal volume dV, is obtained from a Hertzian vector $\underline{Z}$ whose components are given by

$$Z_\alpha = dV \ \tau(t-r/c, x, y, z) \ E_\alpha(t-r/c, x, y, z)/4\pi r \tag{1}$$

The electric field components at x',y',z' are derived from the vector $\underline{Z}$ by the equation,

$$E_\alpha = -\frac{1}{c^2}\frac{\partial^2 Z_\alpha}{\partial t^2} + \sum_\beta \frac{\partial^2 Z_\beta}{\partial x'_\alpha \partial x'_\beta} \tag{2}$$

In a homogeneous medium (the case considered by Darwin), τ is a constant equal to τ_0, independent of space and time coordinates, and is given by §1 (3). However, in a heterogeneous medium, e.g., a liquid traversed by ultrasonic waves, τ will no longer be a constant but will be a function of t, x, y, and z. We assume, in view of the arguments given in the Introduction, that in such a disturbed medium τ (t,x,y,z) is directly proportional to the density of the medium. Then a harmonic compression wave travelling along the x axis gives a scattering index of the form

$$\begin{aligned}\tau(t,x) &= \tau_0\left(1 + \Delta \cos(\Omega t - Kx)\right) \\ &= \tau_0\left(1 + \frac{\Delta}{2} e^{i(\Omega t - Kx)} + \frac{\Delta}{2} e^{-i(\Omega t - Kx)}\right)\end{aligned} \tag{3}$$

where τ_0 is the scattering index of the undisturbed

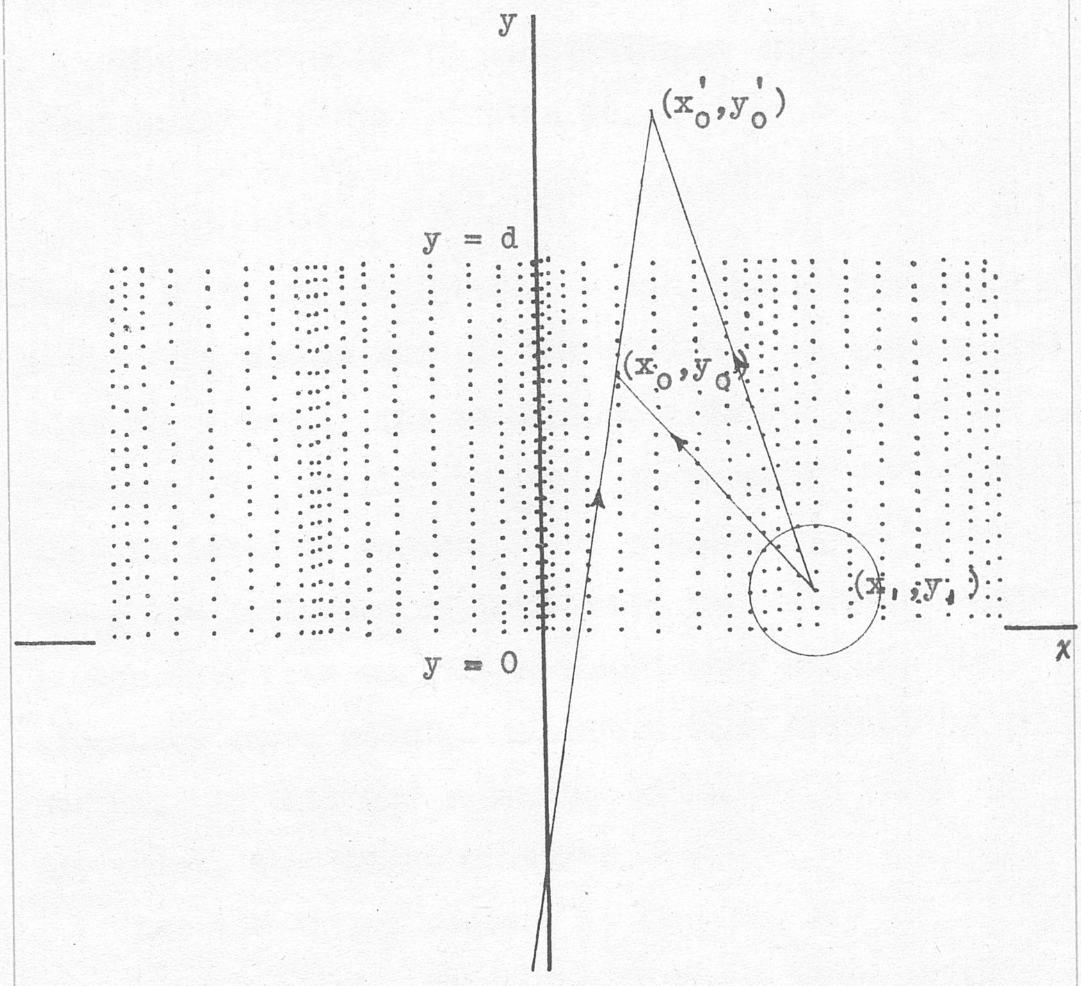

FIGURE 1.

The Ultrasonic Beam as a Scattering Medium.

Single Scattering centre of medium (x_1, y_1)

Centre of integration inside medium (x_0, y_0)

Centre of integration beyond medium (x_0', y_0')

medium, Δ is the ratio of the amplitude of the density wave to the average density in the medium, and Ω is the angular frequency and K the wave number of the ultrasonic disturbance.

We shall confine our attention to progressive ultrasonic waves. Standing waves could, however, be treated in a similar manner using, for τ, in place of (3), the expression

$$\begin{aligned} \tau(t,x) &= \tau_0 (1 + \Delta \cos \Omega t \cos Kx) \\ &= \tau_0 \Big(1 + \frac{\Delta}{4} e^{i(\Omega t - Kx)} + \frac{\Delta}{4} e^{-i(\Omega t - Kx)} \\ &\qquad + \frac{\Delta}{4} e^{i(\Omega t + Kx)} + \frac{\Delta}{4} e^{-i(\Omega t + Kx)} \Big). \end{aligned} \tag{4}$$

2.2 The Integral Equation for the E Polarization.

We consider a disturbed region of liquid lying between the planes, y = 0 and y = d, and extending indefinitely in the x and z directions, (see Figure 1). An ultrasonic wave travels along the x axis and fills the whole region. This slab of liquid is illuminated by light falling on it obliquely with the wave normal lying in the xy-plane and making an angle θ $(-\pi < \theta \leqslant \pi)$ with the y axis. We measure θ anticlockwise from the positive y direction to the direction along which the light advances. We consider first the E polarization, the case of light polarized with the electric vector along the z axis (i.e., at right angles to the plane of incidence).

The equation of an incident wave propagated at right angles to the z axis is

$$E_z(t, x, y) = B e^{i(\omega t - xk\sin\theta - yk\cos\theta)}, \tag{1}$$

where B is the amplitude, ω the angular frequency, k the wave number and θ the angle of the wave normal with the y axis. (As is customary in the use of a function of a complex variable in physics, the real part is taken to represent the physical quantity). Under the influence of this wave, each particle emits a secondary wave and the incident wave and all the secondary waves combine to affect each element of the medium. We call the resulting stimulating force at any point, the "light vector".

Let the "light vector" at (x_1, y_1, z_1) be $\underline{E}(t, x_1, y_1, z_1)$. Since the incident light is polarized along the z axis and our system has the same properties in all planes at right angles to the z axis, it is reasonable to assume that the light vector $\underline{E}(t,x,y,z)$ will also be polarized along the z axis . This light vector causes the emission of a scattered wave and the effect, at the point (x_o, y_o, z_o) and at time t , is given by the Hertzian vector of the form (cf. equation 2.1, (1))

$$dV \; \tau(t-\tfrac{r}{c}, x_1)\, E_z(t-\tfrac{r}{c}, x_1, y_1, z_1)/4\pi r, \tag{2}$$

where

$$r^2 = (x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2.$$

Moreover, in view of the above remarks, we need consider only the z component of the light vector scattered from the volume dV. This is given by

$$dV\left\{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z_0^2}\right\}\tau(t-\tfrac{r}{c}, x_1)\, E_z(t-\tfrac{r}{c}, x_1, y_1, z_1)/4\pi r. \tag{3}$$

The total effect at (x_o, y_o, z_o) is obtained by integrating over the whole volume and adding the incident wave. We then obtain for $E_z(t,x,y,z)$, at any point (x_o, y_o, z_o) in the scattering medium, the integral equation

$$E_z(t, x_0, y_0, z_0) = B\, e^{i(\omega t - x_0 k \sin\theta - y_0 k \cos\theta)}$$

$$+ \iiint dx_1\, dy_1\, dz_1 \left\{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z_0^2}\right\}\tau(t-\tfrac{r}{c}, x_1)\, E_z(t-\tfrac{r}{c}, x_1, y_1, z_1)/4\pi r. \tag{4}$$

Once the light vector is known inside the scattering medium, $E_z(t,x,y,z)$ at any point (x_0', y_0', z_0') outside the medium, is simply the right hand side of equation (4). We now proceed to solve this equation for E_z.

2.3 The Evaluation of a Certain Integral.

Before we solve and discuss the integral equation (4), it will be convenient first to evaluate a certain integral which is needed for its solution.

The integral $I(\omega, a, b)$ is defined by

$$I(\omega, a, b)\, e^{i(\omega t - a x_0 - b y_0)}$$

$$= \iiint dx_1 dy_1 dz_1 \left\{ -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z_0^2} \right\} e^{i(\omega(t - \frac{r}{c}) - a x_1 - b y_1)} / 4\pi r$$

$$= \iiint dx_1 dy_1 dz_1\, e^{i(\omega t - a x_1 - b y_1)} \left\{ \frac{\omega^2}{c^2} + \frac{\partial^2}{\partial z_0^2} \right\} e^{-i\omega r/c} / 4\pi r. \quad (1)$$

Making the substitution $x = x_1 - x_0$, $y = y_1 - y_0$ and $z = z_1 - z_0$, (1) can be rewritten in the form

$$I(\omega, a, b) = I_1 + I_2, \quad (2)$$

where

$$I_1 = \iiint dx\, dy\, dz\, e^{-i(ax + by)} \left(\frac{\partial^2}{\partial z^2} \right) e^{-i\omega r/c} / 4\pi r, \quad (3)$$

and

$$I_2 = \iiint dx\, dy\, dz\; e^{-i(ax+by)} \left(\frac{\omega^2}{c^2}\right) e^{-i\omega r/c}/4\pi r. \qquad (4)$$

We first evaluate the integral I_1 with the help of Green's theorem. We observe that the integrand has a singularity at the centre ($r = 0$) of integration. Since for the validity of Green's theorem, the integrand must be regular throughout the domain of integration, we must exclude this particular point. This is done, in the usual manner, by surrounding the point (x_0, y_0, z_0) by a small sphere of radius ε, integrating throughout the volume outside this sphere and, finally, letting $\varepsilon \to 0$.

Let γ_x, γ_y, γ_z be the direction cosines of the normal drawn inward toward the volume of integration. By using Green's theorem, we can transform the volume integral I_1 into a surface integral and obtain

$$I_1 = \iiint dx\, dy\, dz\; e^{-i(ax+by)} \frac{\partial^2}{\partial z^2}\left(e^{-i\omega r/c}/4\pi r\right)$$

$$= -\iint dS\; \gamma_z\, e^{-i(ax+by)} \frac{\partial}{\partial z}\left(e^{-i\omega r/c}/4\pi r\right)$$

In evaluating this surface integral, we have $\gamma_z = 0$ over the faces of the slab and $\gamma_z = z/r$ over the small sphere about the origin. As we let $\varepsilon \to 0$, we keep only the first order terms and have

$$I_1 = -\iint dS' \frac{z}{r} \frac{\partial^2}{\partial z}\left(\frac{1}{4\pi r}\right) = \iint dS' \, z^2/4\pi r^4 = \frac{1}{3} \qquad (5)$$

For the integral I_2 we require a particular result discussed in detail by Darwin. In brief, he shows that the integral

$$\iint_{-\infty}^{\infty} dx\, dz\, e^{-ik(r+ax)}/4\pi r$$

(where r is positive, y $(\equiv \sqrt{r^2-x^2-z^2})$ is real, and a is real and $|a|$ less then unity) is summable though not strictly convergent. In this summation certain oscillating terms from infinity are rejected on the grounds that to include them would be, in effect, to make a study of the diffraction pattern produced at the centre of the slab by its remote edges. (The presence of finite slit width is not, in general, apparent in ultrasonic diffraction experiments, but when it does appear, as in some plates by Karff (1936), it is easily distinguishable. Darwin finds that, subject to the above conditions,

$$\iint_{-\infty}^{\infty} dx\, dz\, e^{-ik(r+ax)}/4\pi r = e^{-ik|y|\sqrt{1-a^2}}/2ik\sqrt{1-a^2}$$

and we have

$$\iint_{-\infty}^{\infty} dx\, dz\, e^{-i\left(\frac{\omega r}{c} + ax\right)}/4\pi r = e^{-i|y|\sqrt{\frac{\omega^2}{c^2} - a^2}} / 2i\sqrt{\frac{\omega^2}{c^2} - a^2}. \quad (6)$$

Now by writing I_2 as the sum of two integrals and using (6), we obtain

$$I_2 = \frac{\omega^2}{c^2}\left[\int_{-y_0}^{0} + \int_{0}^{d-y_0}\right] dy\, e^{-iby} \iint_{-\infty}^{\infty} dx\, dz\, e^{-i\left(\frac{\omega r}{c} + ax\right)}/4\pi r$$

$$= \frac{\omega^2}{c^2}\int_{-y_0}^{0} dy\, e^{-iby}\, e^{+iy\sqrt{\frac{\omega^2}{c^2} - a^2}} / 2i\sqrt{\frac{\omega^2}{c^2} - a^2}$$

$$+ \frac{\omega^2}{c^2}\int_{0}^{d-y_0} dy\, e^{-iby}\, e^{-iy\sqrt{\frac{\omega^2}{c^2} - a^2}} / 2i\sqrt{\frac{\omega^2}{c^2} - a^2}$$

$$= \frac{\omega^2}{c^2}\left[\frac{1}{\left(b - \sqrt{\frac{\omega^2}{c^2} - a^2}\right)\left(b + \sqrt{\frac{\omega^2}{c^2} - a^2}\right)} - \frac{e^{i\left(b - \sqrt{\frac{\omega^2}{c^2} - a^2}\right) y_0}}{2\left(b - \sqrt{\frac{\omega^2}{c^2} - a^2}\right)\left(\sqrt{\frac{\omega^2}{c^2} - a^2}\right)}\right.$$

$$\left. + \frac{e^{i\left(b + \sqrt{\frac{\omega^2}{c^2} - a^2}\right)(y_0 - d)}}{2\left(b + \sqrt{\frac{\omega^2}{c^2} - a^2}\right)\left(\sqrt{\frac{\omega^2}{c^2} - a^2}\right)}\right] \quad (7)$$

Adding (5) and (7) and using (2), we have finally

$$I(\omega, a, b) = \frac{1}{\sigma(\omega, a, b)} - \frac{\omega^2}{c^2}\, \frac{e^{i g(\omega, a, b) y_0}}{2 g(\omega, a, b)\sqrt{\frac{\omega^2}{c^2} - a^2}}$$

$$+ \frac{\omega^2}{c^2}\, \frac{e^{i h(\omega, a, b)(y_0 - d)}}{2 h(\omega, a, b)\sqrt{\frac{\omega^2}{c^2} - a^2}}, \quad (8)$$

where

$$\left.\begin{aligned} \sigma(\omega, a, b) &= \frac{3\,(a^2 + b^2 - \omega^2/c^2)}{(a^2 + b^2 + 2\omega^2/c^2)}, \\ g(\omega, a, b) &= b - \sqrt{\frac{\omega^2}{c^2} - a^2}, \\ h(\omega, a, b) &= b + \sqrt{\frac{\omega^2}{c^2} - a^2}. \end{aligned}\right\} \quad (9)$$

Having evaluated the integral $I(\omega, a, b)$, we can resume the discussion of the integral equation.

2.4 The Solution of the Integral Equation.

Since $E_z(t,x,y,z)$ is the same for all values of z, we write, for $E_z(t,x,y)$, as a trial solution for our integral equation,

$$E_z(t, x, y) = \sum N_{\ell m}\, e^{i(\omega_{\ell m} t - p_\ell x - q_m y)}, \qquad (1)$$

where ℓ and m are integers ($+$ve or $-$ve), $\omega_{\ell m}$ is the angular frequency, $N_{\ell m}$ the amplitude and we call p_ℓ and q_m the x and y components of a propagation vector. Each term of (1) which contributes to the solution is characterized by two indices, so that the trial solution represents a doubly infinite sheaf of plane waves. This form of a possible solution is suggested by the multiple reflections and refractions to be expected in an infinite slab of stratified medium with parallel plane faces.

In a homogeneous medium with a given refractive index n, corresponding to a given frequency ω, p_ℓ and q_m satisfy the relation

$$\frac{n^2\omega^2}{c^2} = p_\ell^2 + q_m^2 \tag{2}$$

so that for a given frequency and a given value of p_ℓ, there can exist only one value of q_m^2. However, in a stratified medium, as we shall see below, q_m^2 can have an infinite set of values since the refractive index is not constant in the medium +.

We now substitute our trial solution (1) into the integral equation §2.2 (4). Remembering that the scattering index τ is real, we have

$$\tau(t,x_1)E_z(t,x_1,y_1,z_1) = \tau_0 \sum N_{\ell m} e^{-iq_m y_1}\Big[e^{i(\omega_{\ell m}t - p_\ell x_1)} + \frac{\Delta}{2} e^{i((\omega_{\ell m}+\Omega)t - (p_\ell+K)x_1)} + \frac{\Delta}{2} e^{i((\omega_{\ell m}-\Omega)t - (p_\ell-K)x_1)}\Big]. \tag{3}$$

\+ Corresponding to each value of q_m^2, there are, of course two values of q_m viz. $\pm q_m$. The amplitudes of the corresponding solutions will be denoted by $N^+_{\ell m}$ and $N^-_{\ell m}$. It is to be noticed that the sum in (1) includes both kinds of terms, although for the sake of brevity we have not written this explicitly. In future, too, the summation over m would imply summation over both $N^+_{\ell m}$ and $N^-_{\ell m}$ unless otherwise stated.

Writing $x = x_1 - x_0$, $y = y_1 - y_0$, $z = z_1 - z_0$, and using the notation of §2.3, the integral equation reduces to

$$\sum N_{\ell m}\, e^{i(\omega_{\ell m} t - p_\ell x_0 - q_m y_0)} = B\, e^{i(\omega t - x_0 k \sin\theta - y_0 k \cos\theta)}$$

$$+ \tau_0 \sum N_{\ell m}\, e^{i(\omega_{\ell m} t - p_\ell x_0 - q_m y_0)}\; I(\omega_{\ell m}, p_\ell, q_m) \qquad (4)$$

$$+ \frac{\Delta}{2} \tau_0 \sum N_{\ell m}\, e^{i((\omega_{\ell m} + \Omega) t - (p_\ell + K) x_0 - q_m y_0)}\; I\big((\omega_{\ell m} + \Omega), (p_\ell + K), q_m\big)$$

$$+ \frac{\Delta}{2} \tau_0 \sum N_{\ell m}\, e^{i((\omega_{\ell m} - \Omega) t - (p_\ell - K) x_0 - q_m y_0)}\; I\big((\omega_{\ell m} - \Omega), (p_\ell - K), q_m\big).$$

The integral $I(\omega, a, b)$ has been evaluated in §2.3. Substituting from §2.3 (8) into (4) ~~and (5)~~, we obtain finally

$$\sum N_{\ell m}\, e^{i(\omega_{\ell m} t - p_\ell x_0 - q_m y_0)} = B\, e^{i(\omega t - x_0 k \sin\theta - y_0 k \cos\theta)}$$

$$+ \tau_0 \sum N_{\ell m} \Big[e^{i(\omega_{\ell m} t - p_\ell x_0 - q_m y_0)} / \sigma(\omega_{\ell m}, p_\ell, q_m)$$

$$+ \frac{\Delta}{2}\, e^{i((\omega_{\ell m} + \Omega) t - (p_\ell + K) x_0 - q_m y_0)} / \sigma(\omega_{\ell m} + \Omega, p_\ell + K, q_m)$$

$$+ \frac{\Delta}{2}\, e^{i((\omega_{\ell m} - \Omega) t - (p_\ell - K) x_0 - q_m y_0)} / \sigma(\omega_{\ell m} - \Omega, p_\ell - K, q_m)$$

$$
\begin{aligned}
&- i_0 \sum N_{em} \Big[\omega_{em}^2 \, e^{i\left(\omega_{em} t - p_e x_0 - \sqrt{\frac{\omega_{em}^2}{c^2} - p_e^2}\, y_0\right)} / 2c^2 g(\omega_{em}, p_e, q_m) \sqrt{\frac{\omega_{em}^2}{c^2} - p_e^2} \\
&+ \frac{\Delta}{2} (\omega_{em} + \Omega)^2 \frac{e^{i\left((\omega_{em}+\Omega)t - (p_e+K)x_0 - \sqrt{\frac{(\omega_{em}+\Omega)^2}{c^2} - (p_e+K)^2}\, y_0\right)}}{2c^2 g(\omega_{em}+\Omega, p_e+K, q_m) \sqrt{\frac{(\omega_{em}+\Omega)^2}{c^2} - (p_e+K)^2}} \\
&+ \frac{\Delta}{2} (\omega_{em} - \Omega)^2 \frac{e^{i\left((\omega_{em}-\Omega)t - (p_e-K)x_0 - \sqrt{\frac{(\omega_{em}-\Omega)^2}{c^2} - (p_e-K)^2}\, y_0\right)}}{2c^2 g(\omega_e-\Omega, p_e-K, q_m) \sqrt{\frac{(\omega_{em}-\Omega)^2}{c^2} - (p_e-K)^2}} \Big] \qquad (5) \\
&+ i_0 \sum N_{em} \Big[\omega_{em}^2 \frac{e^{i\left(\omega_{em} t - p_e x_0 + \sqrt{\frac{\omega_{em}^2}{c^2} - p_e^2}\, y_0\right)} e^{-ih(\omega_{em}, p_e, q_m) d}}{2c^2 h(\omega_{em}, p_e, q_m) \sqrt{\frac{\omega_{em}^2}{c^2} - p_e^2}} \\
&+ \frac{\Delta}{2} (\omega_{em} + \Omega)^2 \frac{e^{i\left((\omega_{em}+\Omega)t - (p_e+K)x_0 + \sqrt{\frac{(\omega_{em}+\Omega)^2}{c^2} - (p_e+K)^2}\, y_0\right)} e^{-ih(\omega_{em}+\Omega, p_e+K, q_m) d}}{2c^2 h(\omega_{em}+\Omega, p_e+K, q_m) \sqrt{\frac{(\omega_{em}+\Omega)^2}{c^2} - (p_e+K)^2}} \\
&+ \frac{\Delta}{2} (\omega_{em} - \Omega)^2 \frac{e^{i\left((\omega_{em}-\Omega)t - (p_e-K)x_0 + \sqrt{\frac{(\omega_{em}-\Omega)^2}{c^2} - (p_e-K)^2}\, y_0\right)} e^{-ih(\omega_{em}-\Omega, p_e-K, q_m) d}}{2c^2 h(\omega_{em}-\Omega, p_e-K, q_m) \sqrt{\frac{(\omega_{em}-\Omega)^2}{c^2} - (p_e-K)^2}} \Big]
\end{aligned}
$$

In order that equation (5) may be satisfied at all times and at all points of the scattering medium, the coefficients of each exponential which differs from all the others in any of the variables t, x, y must vanish separately. We observe in (5) that ω_{em} changes in steps of Ω and is always accompanied by a change of p_e in steps of K. The coefficients of y_0 in the various exponentials, however, either remains unchanged (q_m) or is always the same function of corresponding ω's and p's. Hence, we can take

$\omega_{\ell m}$ to depend only on the index ℓ. Moreover, since we can assume, without loss of generality, that $\omega_{(\ell=0)}$ is the frequency of the incident light, we obtain

$$\left.\begin{array}{ll} \omega_{\ell=0} = \omega & \omega_\ell = \omega + \ell\Omega \\ p_0 = k\sin\theta & p_\ell = k\sin\theta + \ell K \end{array}\right\} \qquad (6)$$

Using the relations (6), equation (5) can be written as

$$\begin{aligned}
&\sum N_{\ell m}\, e^{i(\omega_{\ell m}t - p_\ell x_0 - q_m y_0)} = B\, e^{i(\omega t - x_0 k\sin\theta - y_0 k\cos\theta)} \\
&+ \tau_0 \sum N_{\ell m}\Big[e^{i(\omega_{\ell m}t - p_\ell x_0 - q_m y_0)}/\sigma_{\ell m} \\
&+ \tfrac{\Delta}{2}\, e^{i(\omega_{\ell+1}t - p_{\ell+1}x_0 - q_m y_0)}/\sigma_{\ell+1,m} \\
&+ \tfrac{\Delta}{2}\, e^{i(\omega_{\ell-1}t - p_{\ell-1}x_0 - q_m y_0)}/\sigma_{\ell-1,m}\Big] \\
&- \tau_0 \sum N_{\ell m}\Big[\omega_\ell^2 e^{i(\omega_\ell t - p_\ell x_0 - \sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2}\, y_0)}/2c^2 g_{\ell m}\sqrt{\tfrac{\omega_\ell^2}{c^2} - p_\ell^2} \\
&+ \tfrac{\Delta}{2}\,\omega_{\ell+1}^2\, e^{i(\omega_{\ell+1}t - p_{\ell+1}x_0 - \sqrt{\frac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2}\, y_0)}/2c^2 g_{\ell+1,m}\sqrt{\tfrac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2} \qquad (7)\\
&+ \tfrac{\Delta}{2}\,\omega_{\ell-1}^2\, e^{i(\omega_{\ell-1}t - p_{\ell-1}x_0 - \sqrt{\frac{\omega_{\ell-1}^2}{c^2} - p_{\ell-1}^2}\, y_0)}/2c^2 g_{\ell-1,m}\sqrt{\tfrac{\omega_{\ell}^2}{c^2} - p_{\ell-1}^2}\Big] \\
&+ \tau_0 \sum N_{\ell m}\Big[\omega_\ell^2\, e^{i(\omega_\ell t - p_\ell x_0 + \sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2}\, y_0)}\, e^{ih_{\ell m}d}/2c^2 h_{\ell m}\sqrt{\tfrac{\omega_\ell^2}{c^2} - p_\ell^2} \\
&+ \tfrac{\Delta}{2}\,\omega_{\ell+1}^2\, e^{i(\omega_{\ell+1}t - p_{\ell+1}x_0 + \sqrt{\frac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2}\, y_0)}\, e^{ih_{\ell+1,m}d}/2c^2 h_{\ell+1,m}\sqrt{\tfrac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2} \\
&+ \tfrac{\Delta}{2}\,\omega_{\ell-1}^2\, e^{i(\omega_{\ell-1}t - p_{\ell-1}x_0 + \sqrt{\frac{\omega_{\ell-1}^2}{c^2} - p_{\ell-1}^2}\, y_0)}\, e^{ih_{\ell-1,m}d}/2c^2 h_{\ell-1,m}\sqrt{\tfrac{\omega_{\ell-1}^2}{c^2} - p_{\ell-1}^2}\Big]
\end{aligned}$$

where

$$\left.\begin{aligned} \sigma_{\ell m} &= \frac{3\,(p_\ell^2 + q_m^2 - \omega_\ell^2/c^2)}{(p_\ell^2 + q_m^2 + 2\,\omega_\ell^2/c^2)} \\ g_{\ell m} &= q_m - \sqrt{\omega_\ell^2/c^2 - p_\ell^2} \\ h_{\ell m} &= q_m + \sqrt{\omega_\ell^2/c^2 - p_\ell^2} \end{aligned}\right\} \quad (8)$$

We mention here that the last six lines of (7) describe the light waves as though they were in vacuum, i.e. to a given frequency ω_ℓ there corresponds a wave number ω_ℓ/c. For this reason, we call them "external waves" although they are <u>actually</u> found inside the scattering medium. We shall see in § 2.5 that outside the scattering medium the solution consists of only these types of terms. The components of the propagation vector in these terms, as we have seen, are independent of the particular value of q_m. The amplitude of each of these waves, however, does depend on the amplitudes $N_{\ell m}$ and on the possible q_m values of our trial solution. We shall now set up the equations for determining $N_{\ell m}$ and the possible values of q_m by setting the coefficients of each of the exponentials separately equal to zero.

Now, it is easy to see that the q_m of our trial solution (1) cannot take the values $\sqrt{\omega_\ell^2/c^2 - p_\ell^2}$

for, if $q_m^2 = \sqrt{\omega_\ell^2/c^2 - p_\ell^2}$ then $\sigma_{\ell m} = 0$ and the right hand side of (7) becomes infinitely great while the left hand side is finite. This is to be expected since our trial solution is in a scattering medium whose refractive index is not unity and the frequency to wave number ratio will not equal the velocity of light in vacuum as a relation of the type $q_m^2 = \sqrt{\omega_\ell^2/c^2 - p_\ell^2}$ implies. Equating first the coefficients of each exponential in (7) which has the factor $q_m y_0$ in it we obtain, for each m the following recurrence relation for the amplitudes

$$N_{\ell m}^{\pm} \left(\sigma_{\ell m}/\tau_0 - 1 \right) = \frac{\Delta}{2} \left(N_{\ell+1,m}^{\pm} + N_{\ell-1,m}^{\pm} \right) \qquad (9)$$

where each relation contains only either N^{+}_{ℓ} or N^{-}_{ℓ}. Similarly from a comparison of the coefficients of other exponentials of (7), we obtain

$$B = \sum_m \tau_0 \, \omega_0^2 \left(N_{om} + \frac{\Delta}{2} N_{1m} + \frac{\Delta}{2} N_{-1,m} \right) / 2c^2 g_{om} \sqrt{\omega_0^2/c^2 - p_0^2}.$$

Using the recurrence relation (9), this simplifies to

$$B = \sum_m \sigma_{om} \, \omega_0^2 \, N_{om} / 2c^2 g_{om} \sqrt{\omega_0^2/c^2 - p_0^2}. \qquad (10)$$

Likewise, we have for the other exponentials

$$0 = \sum_m \sigma_{\ell m} \, \omega_\ell^2 \, N_{\ell m} / 2c^2 g_{\ell m} \sqrt{\omega_\ell^2/c^2 - p_\ell^2} \qquad \text{for } \ell \neq 0, \quad (11)$$

$$0 = \sum_m \sigma_{\ell m} \, \omega_\ell^2 \, N_{\ell m} \, e^{i h_{\ell m} d} / 2c^2 h_{\ell m} \sqrt{\omega_\ell^2/c^2 - p_\ell^2} \qquad \text{for all } \ell. \quad (12)$$

The summation over m in (10), (11) and (12) implies summation over both the N^+'s and N^-'s .

In order to see that the number of equations (9) to (12) is enough to obtain possible values of q_m and for the solution of all the amplitudes $N^{\pm}_{\ell m}$, we consider first the set of recurrence relations (9). Substituting for $\sigma_{\ell m}$ from (8) in (9), we have

$$\left[1 - \frac{(n^2+2)\,(p_\ell^2 + q_m^2 - \omega_\ell^2/c^2)}{(n^2-1)\,(p_\ell^2 + q_m^2 + 2\omega_\ell^2/c^2)}\right] N^{\pm}_{\ell m} + \frac{\Delta}{2}\left(N^{\pm}_{\ell+1,m} + N^{\pm}_{\ell-1,m}\right) = 0 \quad (13)$$

The suffix $\pm$ and m in these relations have as yet no significance, since they occur through the possible values of q_m, which are still undetermined. We, therefore, drop these suffix in (13), for the present, in order to introduce them later to label different possible values of q. Doing so, we re-write (13) as

$$f_\ell(q^2)\,N_\ell + \frac{\Delta}{2}\left(N_{\ell+1} + N_{\ell-1}\right) = 0, \quad (14)$$

where, we have written, for brevity, $f_\ell(q^2)$ for the expression within the square brackets in (13),

$$f_\ell(q^2) = 1 - \frac{(n^2+2)\,(p_\ell^2 + q_m^2 - \omega_\ell^2/c^2)}{(n^2-1)\,(p_\ell^2 + q_m^2 + 2\omega_\ell^2/c^2)}. \quad (15)$$

Now the relations (14) can be regarded as an infinite set of homogeneous simultaneous equations. The condition that (14) may have a non-trivial solution i.e. $N_\ell \neq 0$ for all ℓ, is that the infinite determinant formed from the coefficients of (14) vanishes, i.e.

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & f_{-1}(q^2), & \frac{\Delta}{2}, & 0 & \cdot \\ \cdot & \frac{\Delta}{2}, & f_0(q^2), & \frac{\Delta}{2} & \cdot \\ \cdot & 0, & \frac{\Delta}{2}, & f_1(q^2) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \qquad (16)$$

The roots of this equation determine the possible values of q^2. If Δ were zero, q^2 is just given by

$$q^2 = \frac{n^2\omega_m^2}{c^2} - p_m^2 \qquad (-\infty < m < \infty) \qquad (17)$$

If $\Delta \neq 0$, we denote by the suffix m on q^2, that value of q^2 which reduces to $n^2\omega_m^2/c^2 - p_m^2$ when one puts $\Delta = 0$. Corresponding to each value of q_m^2, there are, of course, two values of q_m, i.e. $\pm q_m$.

The amplitudes $N_{\ell m}$ corresponding to a $+q_m$ and $-q_m$, as mentioned before, will be denoted by $N_{\ell m}^+$ and $N_{\ell m}^-$ respectively. The equations (13), then, for a given value of q, say $+q_m$, will give all amplitudes $N_{\ell m}^+$ $(-\infty < \ell < \infty)$, in terms of one of them which, for convenience, we take N_{mm}^+. In this way, one can obtain with the help of (13) all the amplitudes $N^{\pm}_{\ell m}$, in terms of a single infinity of unknowns N^{+}_{mm} and a single infinity of unknowns N^{-}_{mm}. It is seen that equations (10), (11) and (12) are just enough in number to obtain these unknowns as solution of these equations. We must mention, that in order that the inhomogeneous equations (10) to (12) should have a unique solution, the corresponding determinant must not vanish. This, however, is difficult to prove, and we shall assume here, that this condition is satisfied.

The equations (9) to (12), therefore, determine the possible values of q_m and the infinite set of amplitudes $N^{\pm}_{\ell m}$ $(-\infty < {\ell \atop m} < \infty)$. Although it does not appear possible in general to solve these equations explicitly, approximate solutions can be obtained in any particular case (i.e. for given values of the parameters Δ and α). The methods for obtaining such approximate solutions will be discussed in § 3.1.

§2.5 Diffracted Spectra Outside the Scattering Medium.

We have shown in the previous section that the integral equation for $\underline{E}$ in the disturbed medium can be solved and the solution can be written in the form

$$E_z(t,x,y) = \sum N_{\ell m}\, e^{i(\omega_\ell t - p_\ell x - q_m y)}, \qquad (1)$$

where ω_ℓ and p_ℓ are given by §2.4 (6) and q_m and $N_{\ell m}$ are determined by equations (9) to (12) of §2.4. We now proceed to obtain explicit expressions for the frequencies, directions and amplitudes of the diffracted spectra outside the scattering medium. It follows from §2.2 that the light vector $E_z^0(t, x_0', y_0')$ at a point (x_0', y_0', z_0') outside the medium and at time t is given by

$$E_z^0(t, x_0', y_0') = \iiint dx_1\, dy_1\, dz_1 \left\{ -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z_0^2} \right\} \tau\left(t - \frac{r}{c}, x_1\right) E_z\left(t - \frac{r}{c}, x_1, y_1, z_1\right) \Big/ 4\pi r, \tag{2}$$

where the integration is over the entire scattering medium, (x_1, y_1, z_1) is a point inside this medium, $E_z(t, x_1, y_1, z_1)$ is given by (1), $\tau(t, x_1)$ is given by §2.1 (3) and $r^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2$. When $\tau(t - \frac{r}{c}, x_1)\, E_z(t - \frac{r}{c}, x_1, y_1, z_1)$ is written explicitly, we again have to use the integrals $I(\omega, a, b)$, discussed in §2.3. The evaluation of the integral $I(\omega, a, b)$ is, however, much simpler in the present case since the point (x_0', y_0', z_0') lies outside the region of integration and there is then no singularity (at the origin) of the kind discussed in that section. Hence, the integral I_1 of §2.3 is zero, while I_2, for a point beyond the scattering medium, can be easily seen to be given by

$$I_2 = \frac{\omega^2}{c^2}\left[\left(e^{i g(\omega, a, b) d} - 1\right) \Big/ 2 g(\omega, a, b) \sqrt{\frac{\omega^2}{c^2} - a^2}\,\right] \tag{3}$$

while, for a point before the scattering medium, I_2 is given by

$$I_2 = \frac{\omega^2}{c^2}\left[\left(e^{-ih(\omega,a,b)d} - 1\right) / 2h(\omega,a,b)\sqrt{\frac{\omega^2}{c^2} - a^2}\right] \qquad (4)$$

Making use of (3), we obtain for $E_z^0(t,x_0',y_0')$ beyond the scattering medium

$$E_z^0(t,x_0',y_0') = \tau_0 \Sigma N_{\ell m}\Big[\omega_\ell^2\left(e^{-ig_{\ell m}d} - 1\right) e^{i\left(\omega_\ell t - p_\ell x_0' - \sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2}\, y_0'\right)} / 2c^2 g_{\ell m}\sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2}$$
$$+ \frac{\Delta}{2}\omega_{\ell+1}^2\left(e^{-ig_{\ell+1,m}d} - 1\right) e^{i\left(\omega_{\ell+1}t - p_{\ell+1}x_0' - \sqrt{\frac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2}\, y_0'\right)} / 2c^2 g_{\ell+1,m}\sqrt{\frac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2}$$
$$+ \frac{\Delta}{2}\omega_{\ell-1}^2\left(e^{-ig_{\ell-1,m}d} - 1\right) e^{i\left(\omega_{\ell-1}t - p_{\ell-1}x_0' - \sqrt{\frac{\omega_{\ell-1}^2}{c^2} - p_{\ell-1}^2}\, y_0'\right)} / 2c^2 g_{\ell-1,m}\sqrt{\frac{\omega_{\ell-1}^2}{c^2} - p_{\ell-1}^2}\Big] \qquad (5)$$

Similarly, at a point before the scattering medium, $E_z^0(t,x_0',y_0')$ is given by

$$E_z^0(t,x_0',y_0') = \tau_0 \Sigma N_{\ell m}\Big[\omega_\ell^2\left(e^{-ih_{\ell m}d} - 1\right) e^{i\left(\omega_\ell t - p_\ell x_0' - \sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2}\, y_0'\right)} / 2c^2 h_{\ell m}\sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2}$$
$$+ \frac{\Delta}{2}\omega_{\ell+1}^2\left(e^{-ih_{\ell+1,m}d} - 1\right) e^{i\left(\omega_{\ell+1}t - p_{\ell+1}x_0' - \sqrt{\frac{\omega_{\ell+1}^2}{c^2} - p_{\ell+1}^2}\, y_0'\right)} / 2c^2 h_{\ell+1,m}\sqrt{\frac{\omega_{\ell+1}^2}{c^2} - \ldots}$$
$$+ \frac{\Delta}{2}\omega_{\ell-1}^2\left(e^{-ih_{\ell-1,m}d} - 1\right) e^{i\left(\omega_{\ell-1}t - p_{\ell-1}x_0' - \sqrt{\frac{\omega_{\ell-1}^2}{c^2} - p_{\ell-1}^2}\, y_0'\right)} / 2c^2 h_{\ell-1,m}\sqrt{\frac{\omega_{\ell-1}^2}{c^2} - \ldots} \qquad (6)$$

Since, in general, the intensities of the reflected spectra are very small, we shall confine our attention to points beyond the scattering medium. The total transmitted light is obtained by adding the incident wave to (5). Doing so and making use of the equations §2.4 (10) and (11), we obtain for the transmitted light

$$\sum_l \left[\left(\sum_m \sigma_{lm} N_{lm} \omega_l^2 e^{-ig_{lm}d} \Big/ 2c^2 g_{lm}\sqrt{\frac{\omega_l^2}{c^2} - p_l^2}\right) e^{i\left(\omega_l t - p_l x_0' - \sqrt{\frac{\omega_l^2}{c^2} - p_l^2}\, y_0'\right)} \right. \tag{7}$$

or, writing

$$B_l = \sum_m \sigma_{lm} N_{lm} \omega_l^2 e^{-ig_{lm}d} \Big/ 2c^2 g_{lm}\sqrt{\frac{\omega_l^2}{c^2} - p_l^2}, \tag{8}$$

(7) can be written in the form

$$\sum_l B_l e^{i\left(\omega_l t - p_l x_0' - \sqrt{\frac{\omega_l^2}{c^2} - p_l^2}\, y_0'\right)}. \tag{9}$$

It is immediately seen from (9) that the transmitted wave complex consists of many plane waves, each with a different frequency and a different direction of propagation. Making use of the relations §2.3 (6), the frequencies and the directions of propagation of the various waves are given by

$$\omega_l = \omega + l\Omega \qquad (-\infty < l < \infty) \tag{10}$$

$$\phi_l = \sin^{-1} \frac{p_l c}{\omega_l}$$

$$= \sin^{-1} \frac{(k \sin\theta + lK)c}{\omega + l\Omega} \qquad (11)$$

where ϕ_l denotes the angle between the direction of propagation of the l_{th} wave and the y axis with the same sign convention as for θ. Moreover the amplitude of the l_{th} wave is just B_l which in terms of N_{lm} is given by (8).

If we neglect terms of the order of v/c, where v is the velocity of sound $(v = \Omega/K)$, (11) can be written in the form,

$$\Lambda (\sin\theta - \sin\phi_l) = l\lambda \qquad (-\infty < l < \infty) \qquad (12)$$

These relations for the frequencies and the directions for the various order lines are the same as given by Brillouin and Raman and Nath and, as is well known, are in agreement with the experimental results (cf. Bergmann (1949)). We shall now proceed to discuss the solution of the equations (9) to (12) of §2.4 which enable us then to obtain the intensities of the various orders.

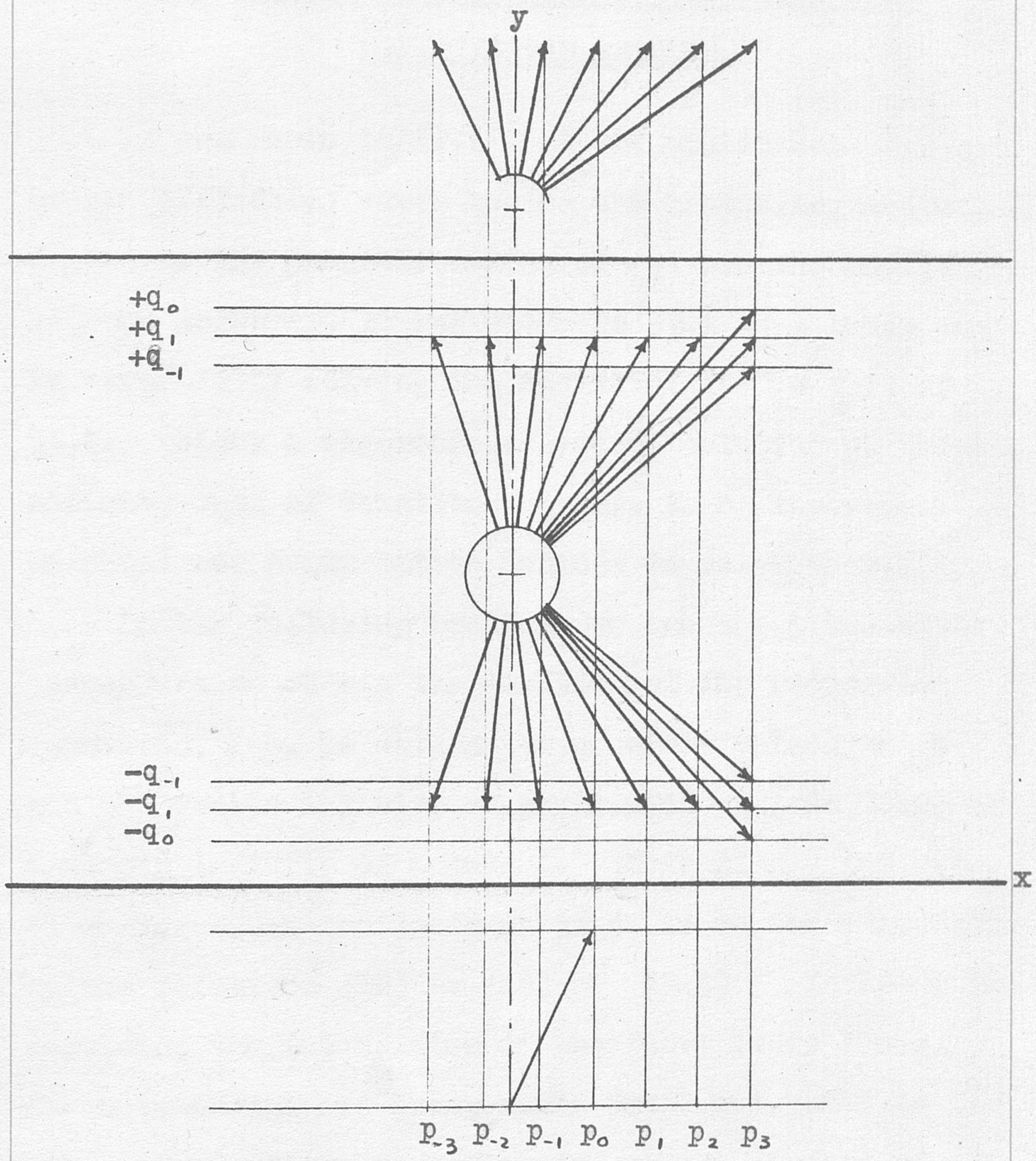

FIGURE 2.

Wave Normals for Incident, Scattered and Diffracted Waves.

III. APPROXIMATE SOLUTIONS AND COMPARISON WITH EXPERIMENTAL RESULTS.

It was shown in §2.5 that the amplitudes B_ℓ, of the diffracted waves beyond the scattering medium, depend on the possible values of q and the amplitudes $N^{\pm}_{\ell m}$. Moreover, it was shown in §2.4 that these can be obtained by solving the equations (9) to (12) of §2.4. Since a rigorous analytical solution of these infinite sets of equations appears to be impossible we shall use perturbation methods to solve them.

In the following section, we use the perturbation procedure to obtain the solution of the recurrence equations, i.e. to obtain the possible values of q and the double infinity of amplitudes $N_{\ell m}$ in terms of a single infinity of unknowns. (This single infinity of unknowns, as mentioned in §2.4 is to be determined by the relations (10) to (12) of §2.4) In later sections, the intensities of the first order lines, for both normal and Bragg angle incidence, will be calculated and the justification of the neglect of certain reflection effects by other workers (cf. Brillouin, (1933)) will be discussed. Finally, we obtain the conditions under which the perturbation procedure is valid and discuss the number of orders likely to appear under various experimental conditions

(§3.4). Moreover, at appropriate places, the results of the theory are compared with experimental findings.

§3.1.1. Perturbation Method of Solution of the Recurrence Relations.

Consider the recurrence relations §2.4 (9) written in their alternate form (cf. §2.4 equation (14)),

$$f_\ell(q^2)\, N_\ell = \frac{\Delta}{2}\left(N_{\ell-1} + N_{\ell+1}\right) \qquad (-\infty < \ell < \infty) \tag{1}$$

where

$$f_\ell(q^2) = \frac{(p_\ell^2 + q^2 - \omega_\ell^2/c^2)(n^2+2)}{(p_\ell^2 + q^2 + 2\omega_\ell^2/c^2)(n^2-1)} - 1. \tag{2}$$

(In future, for the sake of convenience, we shall write η for q^2). Regarding $\Delta/2$ (Δ is in general of the order of 10^{-4}) as a small parameter in (1) and following the usual perturbation procedure, one can expand η (i.e. q^2) and the amplitudes N_ℓ in powers of $\Delta/2$. Doing so, we write

$$N_\ell = N_\ell^{0} + \frac{\Delta}{2} N_\ell^{(1)} + \left(\frac{\Delta}{2}\right)^2 N_\ell^{(2)} + \cdots \tag{3}$$

$$\eta = \eta^{0} + \frac{\Delta}{2}\eta^{(1)} + \left(\frac{\Delta}{2}\right)^2 \eta^{(2)} + \cdots \tag{4}$$

Making use of (4), $f_\ell(\eta)$ can be written as

$$f_\ell(\eta) = f_\ell(\eta^0) + \frac{\Delta}{2}\eta^{(1)} f_\ell'(\eta^0) + \left(\frac{\Delta}{2}\right)^2\left[\eta^{(2)} f_\ell'(\eta^0) + \frac{1}{2}\eta^{(1)} f_\ell''(\eta^0)\right] + \cdots \tag{5}$$

where a dash on f denotes the differential coefficient of f with respect to η. Substituting (3), (4) and (5) in (1), we obtain

$$\left[f_\ell(\eta^0) + \frac{\Delta}{2}\eta^{(1)} f_\ell'(\eta^0) + \left(\frac{\Delta}{2}\right)^2 \left\{\eta^{(2)} f_\ell'(\eta^0) + \frac{1}{2}\eta^{(1)} f_\ell''(\eta^0)\right\} + \cdots\right]$$

$$\left[N_\ell^0 + \frac{\Delta}{2} N_\ell^{(1)} + \left(\frac{\Delta}{2}\right)^2 N_\ell^{(2)} + \cdots\right] = \frac{\Delta}{2}\left[N_{\ell-1}^0 + \cdots \quad N_{\ell+1}^0 + \cdots\right]. \tag{6}$$

Now, in order to get the amplitudes and η correct to various orders in $\Delta/2$, one equates to zero the coefficient of each power of $\Delta/2$ in (6) separately. Putting the terms which are independent of Δ equal to zero, we obtain, in the <u>zeroth approximation</u>,

$$f_\ell(\eta^0) \quad N_\ell^0(\eta) = 0 \tag{7}$$

where $N_\ell^0(\eta)$ indicates the amplitude of that wave whose frequency is characterized by the subscript ℓ and whose q^2 equals η. Equation (7) has either the solution

$$f_\ell(\eta^0) = 0 \qquad\qquad N_\ell^0 \neq 0 \tag{8}$$

or the solution

$$N_\ell = 0 \qquad\qquad f_\ell(\eta^0) \neq 0 \tag{9}$$

Denoting the value of η^0 as given by (8) by the subscript ℓ, (8) and (9) give

$$\eta_l^0 = \frac{n^2\omega_e^2}{c^2} - p_e^2 , \qquad N_l^0(\eta_l^0) \neq 0 \tag{10}$$

and

$$N_{l'}^0(\eta_l^0) \equiv N_{l'l} = 0 \qquad \text{for } l' \neq l \tag{11}$$

Equation (11), of course, holds only if $f_{l'}(\eta_l^0) \neq 0$ for all values $l' \neq l$. We shall call this case non-degenerate to distinguish it from the degenerate case for which there exist more than one value of l ($l_1, l_2, \ldots$ say) such that the equations

$$f_{l_1}(\eta^0) = 0 , \qquad f_{l_2}(\eta^0) = 0 \tag{12}$$

have the same root η^0. We shall first consider the non-degenerate case.

§3.1.2. <u>Non-Degenerate Case.</u>

In this case, as we have seen in the preceding section, for a given value of η^0 ($\eta^0 = \eta_m^0$,say), there is only one non-zero amplitude in the zeroth approximation, namely $N_m(\eta_m^0)$. To obtain other $N_l(\eta_m)$ ($\equiv N_{lm}$) for this value of η_m^0, one has to carry out the perturbation calculation to higher orders. To do so, we equate the coefficient of in §3.1.1 (6) to zero and obtain

$$f_l(\eta_m^0)\, N_l^{(1)}(\eta_m^0) + \eta_m^{(1)} f_l'(\eta_m^0)\, N_l^0(\eta_m^0) = N_{l-1}^0(\eta_m^0) + N_{l+1}^0(\eta_m^0). \tag{1}$$

Putting $l = m$ in (1), we find immediately that $\eta_m^{(1)}$ is zero and there is no first order correction to η_m^{0}. Next by putting $l = m + 1$ and $l = m - 1$ successively in (1), we find

$$N^{(1)}_{m+1,m} = \frac{N_{mm}}{f_{m+1}(\eta_m^0)} \qquad N^{(1)}_{m-1,m} = \frac{N_{mm}}{f_{m-1}(\eta_m^0)} \tag{2}$$

Further, it follows from (1) that $N_{lm} = 0$ for all such that $l \leqslant m - 2$ or $l \geqslant m + 2$.

Similarly, by equating to zero the coefficient of $(\Delta/2)^2$ in §3.1.1 (6), one easily obtains

$$\eta_m^{(2)} = \frac{1}{f_m'(\eta_m^0)}\left[\frac{1}{f_{m+1}(\eta_m^0)} + \frac{1}{f_{m-1}(\eta_m^0)}\right] \tag{3}$$

$$N^{(2)}_{m\pm 2,m} = \frac{N_{mm}}{f_{m\pm 1}(\eta_m^0)\, f_{m\pm 2}(\eta_m^0)} \tag{4}$$

while all the higher N_{lm} are zero. One can also obtain second order corrections to $N_{m\pm 1,m}$ in a straight forward manner at this stage of approximation; they turn out to be zero. It will be seen from (3) that $\eta_m^{(2)}$ would become infinite if $f_m'(\eta_m^0)$ were to be zero. This would invalidate the perturbation calculation. $f_m'(\eta_m^0)$, however, is never zero in our case, as can be easily verified by using the expression §3.1.1 (2) for $f_m(\eta)$.

§3.1.3. Degenerate Case.

Let $\eta^{\circ}_{\ell_1}$ be the root of the equation $f_{\ell_1}(\eta^{\circ}_{\ell_1}) = 0$. If there exists another integer , such that $f_{\ell_2}(\eta^{\circ}_{\ell_1}) = 0$, then it follows from §3.1.1 (10) that

$$\frac{n^2\omega_{\ell_2}^2}{c^2} - p_{\ell_2}^2 = \frac{n^2\omega_{\ell_1}^2}{c^2} - p_{\ell_1}^2. \tag{1}$$

Substituting into (1) the ω's and ps from equations (6a) and (6b) of § 2.4 , we obtain, after some straight-forward algebra,

$$\ell_1 = -\ell_2 - \frac{2(\sin\theta - n^2v/c)}{\alpha(1 - n^2v^2/c^2)} \tag{2}$$

Since for all values of α , within the experimental range, $n^2v/c\alpha \sim 10^{-2}$ or less and $v^2/c^2 \sim 10^{-10}$, (2) can be approximated by $\alpha = \frac{K}{R} = \frac{\lambda}{\Lambda}$

$$\ell_1 = -\ell_2 - 2\,\frac{\sin\theta}{\alpha}, \tag{3a}$$

or

$$\ell_2 - \ell_1 = 2\ell_2 + 2\,\frac{\sin\theta}{\alpha}. \tag{3b}$$

From (3), it follows that if ℓ_1 has the special value

$$l_1 = -\frac{\sin\theta}{\alpha} = L \quad , \text{say}, \tag{4}$$

then $l_1 = l_2$. This implies that for this value of Θ there is no degeneracy and the perturbation method given in the previous section applies to this special case for obtaining the amplitudes $N_{L\pm1,L}$, $N_{L\pm2,L}$ etc., in terms of N_{LL}. When, however, $l_1 \neq L$, it is easily seen from (3) and (4) that, for a given value of l_1, there exists one and only one other integer (l_2) such that $\eta^\circ_{l_2} = \eta^\circ_{l_1}$. Physically this means that corresponding to a given value of η° (or more precisely q^0, cf. foot-note p. 18) there exist in the zeroth order two waves of different frequencies and p components. Hence, for this case, we take as the zeroth order solution for N^0

$$N^\circ_l(\eta^\circ) = \delta_{ll_1} N_{l_1}(\eta^\circ) + \delta_{ll_2} N_{l_2}(\eta^\circ) \tag{5}$$

instead of §3.1.1 (10). Here $N^0_{l_1}(\eta^\circ)$ is the amplitude of the wave whose frequency is characterized by the integer l_1 and whose q value is given by the constant η°. Substituting (5) into §3.1.1 (6) and equating the coefficient of $\Delta/2$ to zero, we obtain

$$\left[\delta_{ll_1} N^\circ_l(\eta^\circ) + \delta_{ll_2} N^\circ_l(\eta^\circ)\right] \eta^{(1)} f'_l(\eta^\circ)$$
$$+ N^{(1)}_l(\eta^\circ) f_l(\eta^\circ) = N^\circ_{l+1}(\eta^\circ) + N^\circ_{l-1}(\eta^\circ).$$

By making use of (5), this can be rewritten as

$$\left[\delta_{\ell\ell_1} N_\ell^0(\eta^0) + \delta_{\ell\ell_2} N_\ell^0(\eta^0)\right]\eta^{(1)} f_\ell'(\eta^0) + N_\ell^{(1)}(\eta^0) f_\ell(\eta^0)$$
$$= (\delta_{\ell_1\ell+1} + \delta_{\ell_2\ell+1}) N_{\ell+1}^0(\eta^0) + (\delta_{\ell_1\ell-1} + \delta_{\ell_2\ell-1}) N_{\ell-1}^0(\eta^0). \tag{6}$$

Putting $\ell = \ell_1 \neq \ell_2$ and $\ell = \ell_2 \neq \ell_1$ successively in (6) gives

$$N_{\ell_1}^0(\eta^0)\,\eta^{(1)} f_{\ell_1}'(\eta^0) = \delta_{\ell_2\ell_1+1} N_{\ell_1+1}^0(\eta^0) + \delta_{\ell_2\ell_1-1} N_{\ell_1-1}^0(\eta^0). \tag{7a}$$

$$N_{\ell_2}^0(\eta^0)\,\eta^{(1)} f_{\ell_2}'(\eta^0) = \delta_{\ell_1\ell_2+1} N_{\ell_2+1}^0(\eta^0) + \delta_{\ell_1\ell_2-1} N_{\ell_2-1}^0(\eta^0). \tag{7b}$$

Now, for convenience, it will be assumed that the lesser of the two integers ℓ_1 and ℓ_2 is ℓ_1. Further discussion can be divided into two cases:

Case I $\ell_2 - \ell_1 = +1$

When $\ell_2 - \ell_1 = 1$, equations (7) give

$$\left.\begin{aligned} N_{\ell_1}^0(\eta^0)\,\eta^{(1)} f_{\ell_1}'(\eta^0) &= N_{\ell_2}^0(\eta^0) \\ N_{\ell_2}^0(\eta^0)\,\eta^{(1)} f_{\ell_2}'(\eta^0) &= N_{\ell_1}^0(\eta^0) \end{aligned}\right\} \tag{8}$$

Eliminating $N_{\ell_1}^0$ and $N_{\ell_2}^0$, we obtain

$$\eta^{(1)} = \pm\sqrt{\frac{1}{f_{\ell_1}'(\eta^0)\, f_{\ell_2}'(\eta^0)}} \tag{9}$$

which then gives

$$\frac{N_{\ell_1}^0(\eta^0)}{N_{\ell_2}^0(\eta^0)} = \pm\sqrt{\frac{f_{\ell_2}'(\eta^0)}{f_{\ell_1}'(\eta^0)}} \tag{10}$$

so that there are two solutions for $N_\ell^0(\eta^\circ)$ of (5) corresponding to two linear combinations.

Next for $\ell \neq \ell_1 \neq \ell_2$, (6) reduces to

$$N_\ell^{(1)}(\eta^\circ)\, f_\ell(\eta^\circ) = (\delta_{\ell_1,\ell+1} + \delta_{\ell_2,\ell+1})\, N_{\ell+1}^\circ(\eta^\circ) + (\delta_{\ell_1,\ell-1} + \delta_{\ell_2,\ell-1})\, N_{\ell-1}^\circ(\eta^\circ)$$

This gives

$$\left.\begin{aligned} N_{\ell_2+1}^{(1)}(\eta^\circ) &= N_{\ell_2}^\circ(\eta^\circ)\, \frac{1}{f_{\ell_2+1}(\eta^\circ)} \\ N_{\ell_1-1}^{(1)}(\eta^\circ) &= N_{\ell_1}^\circ(\eta^\circ)\, \frac{1}{f_{\ell_1-1}(\eta^\circ)} \end{aligned}\right\} \qquad (11)$$

while all N_ℓ for which $\ell < \ell_1 - 1$ or $\ell > \ell_2 + 1$ are zero.

This case will be needed when we come to the calculation of the intensities of the first order Bragg reflections (cf. §3.3.3).

Case II $\ell_2 - \ell_1 \geq 2$

In this case, equations (7) give $\eta^{(1)} = 0$ and $N_{\ell_1}^\circ$ and $N_{\ell_2}^\circ$ are independent of each other. Further, from (6), one can easily obtain

$$\left.\begin{aligned} N_{\ell_1-1}^{(1)} &= N_{\ell_1}^\circ(\eta^\circ)\, \frac{1}{f_{\ell_1-1}(\eta^\circ)} \\ N_{\ell_2+1}^{(1)} &= N_{\ell_2}^\circ(\eta^\circ)\, \frac{1}{f_{\ell_2+1}(\eta^\circ)} \end{aligned}\right\} \qquad (12)$$

$$N_\ell^{(1)} = 0 \qquad \text{for all } \ell < \ell_1 - 1 \text{ or } \ell > \ell_2 + 1$$

For ℓ lying between ℓ_1 and ℓ_2, $\ell = \ell_1 + k$, say, $(k < \ell_2 - \ell_1)$, we have from (6)

$$N^{(1)}_{\ell_1+k} = \frac{1}{f_{\ell_1+k}(\eta^0)} \left[\delta_{\ell_2\, \ell_1+k+1} N^0_{\ell_2}(\eta^0) + \delta_{\ell_1, \ell_1+k-1} N^0_{\ell_1}(\eta^0) \right]. \quad (13)$$

Equation (13) gives non-zero values of $N^{(1)}_{\ell_1+k}$ when and only when either $\ell_1 + k = \ell_2 - 1$ or $k = 1$.

At this stage of the perturbation calculation one has to treat the two cases corresponding to $\ell_2 - \ell_1 > 2$ and $\ell_2 - \ell_1 = 2$ separately.

Case IIa $\ell_2 - \ell_1 > 2$

Here we have

$$\left.\begin{aligned} N^{(1)}_{\ell_1+1}(\eta^0) &= \frac{1}{f_{\ell_1+1}(\eta^0)} N^0_{\ell_1}(\eta^0) \\ N^{(1)}_{\ell_2+1}(\eta^0) &= \frac{1}{f_{\ell_2+1}(\eta^0)} N^0_{\ell_2}(\eta^0) \end{aligned}\right\} \quad (14)$$

while, for other values of ℓ lying between ℓ_1 and ℓ_2, $N^{(1)}_{\ell}(\eta^0)$ is zero. Combining the results (12) and (14), we obtain for this case

$$\left.\begin{aligned} N^{(1)}_{\ell_1\pm1}(\eta^0) &= \frac{1}{f_{\ell_1\pm1}(\eta^0)} N^0_{\ell_1}(\eta^0) \\ N^{(1)}_{\ell_2\pm1}(\eta^0) &= \frac{1}{f_{\ell_2\pm1}(\eta^0)} N^0_{\ell_2}(\eta^0) \end{aligned}\right\} \quad (15)$$

while all other N_ℓ's are zero in this approximation.

Case IIb $\ell_2 - \ell_1 = 2$

Here $N_{\ell_1 - 1}$ and $N_{\ell_2 + 1}$ are again given by (12) but

$$N^{(1)}_{\ell_2 - 1}(\eta^\circ) = N^{(1)}_{\ell_1 + 1}(\eta^\circ) = \frac{1}{f_{\ell_1+1}(\eta^\circ)}\left[N^\circ_{\ell_2}(\eta^\circ) + N^\circ_{\ell_1}(\eta^\circ)\right]. \quad (16)$$

while all amplitudes N_ℓ, not comprised in (5), (12) and (16), are zero.

It will be remembered that all $N^{(1)}_\ell$'s given by (11) to (16) are to be multiplied by $\Delta/2$ to obtain the corresponding amplitudes N_ℓ in the first approximation. In an analogous fashion one can proceed to second order perturbation calculation. But, since this calculation is very lengthy, it will not be given here. The validity of the perturbation procedure and the number of orders likely to appear under given experimental conditions will be discussed in §3.4.

§3.2. Example: Derivation of the Well-known Reflectivity Formula, $R = \left(\frac{n-1}{n+1}\right)^2$ for the Limiting Case $\Delta \to 0$.

In this section, as an example, we calculate the ratio between the intensities of light travelling inside a homogeneous plane parallel slab in the direction of the refracted ray and the light (inside the slab) travelling in the direction of the ray reflected from the lower face of the slab. This ratio for light incident on the slab in the direction of the normal to the face is given by (cf. Tolansky, p. 9).

$$R = \left(\frac{n-1}{n+1}\right)^2 \tag{1}$$

In the notation of our theory, it is obvious that this ratio is given by $(N_{oo}^{-}/N_{oo}^{+})^2$. Now for a homogeneous slab ($\Delta = 0$), the zeroth order solution of the recurrence relations corresponds to the exact solution. From §3.1.1, it then follows that the only possible non-zero amplitudes are $N_{oo}^{\pm}$, $N_{11}^{\pm}$ .... $N_{mm}^{\pm}$ ..... . This set of amplitudes is now to be determined by the remaining equations (10) to (12) of §2.4. Making use of the above solution of the recurrence relations and remembering that the g's and h's are given by §2.4 (8), the equations (10) to (12) can be written as

$$\frac{2c^2 s_\ell}{\sigma_{\ell\ell}\,\omega_\ell^2}\,\delta_{\ell o}\,B = \frac{N_{\ell\ell}^{+}}{q_\ell - s_\ell} + \frac{N_{\ell\ell}^{-}}{-q_\ell - s_\ell} \tag{2a}$$

$$0 = \frac{N_{\ell\ell}^{+}\,e^{i(q_\ell + s_\ell)d}}{q_\ell + s_\ell} + \frac{N_{\ell\ell}^{-}\,e^{-i(q_\ell - s_\ell)d}}{-q_\ell + s_\ell} \tag{2b}$$

for all +ve and integer ℓ's. Here q_ℓ and s_ℓ are given by

$$\left.\begin{aligned} q_\ell &= +\sqrt{\frac{n^2\omega_\ell^2}{c^2} - p_\ell^2} \\ s_\ell &= +\sqrt{\frac{\omega_\ell^2}{c^2} - p_\ell^2} \end{aligned}\right\} \qquad (3)$$

The two sets of equations (2a) and (2b), it will be recalled, are to be interpreted as four sets of equations since the equality sign implies that the real and imaginary parts of the right hand side and the left hand side of each of these equations are identical separately. Now it is easy to verify, from these equations, that $N^{+}_{\ell\ell}$ and $N^{-}_{\ell\ell}$ are zero except when $\ell = 0$ as is to be expected since the light, on being refracted in a homogeneous medium, will not change its frequency. From the equation (26) for $\ell = 0$, we have

$$\frac{N^{-}_{00}}{N^{+}_{00}} = \frac{-q_0 + s_0}{q_0 + s_0}\, e^{-2iq_0 d} \qquad (4)$$

Making use of (3) and remembering that $p_0 = k \sin\theta$, we get from (4)

$$\left|\frac{N^{-}_{00}}{N^{+}_{00}}\right|^2 = \left(\frac{\sqrt{n^2 - \sin^2\theta} - \cos\theta}{\sqrt{n^2 - \sin^2\theta} + \cos\theta}\right)^2 = R', \text{ say,} \qquad (5)$$

which, for normal incidence, reduces to

$$R'_{\text{normal}} = R = \left|\frac{N^{-}_{00}}{N^{+}_{00}}\right|^2 = \left(\frac{n-1}{n+1}\right)^2, \qquad (6)$$

which verifies our theory for this simple case.

Before closing this section, we shall give an explicit expression for N_{oo}^{+}, since it will be useful for some of the discussions in the following paragraphs. Making use of (4) and (2a), we can write for N_{oo}^{+}, after some straightforward algebra,

$$N_{oo}^{+} = \frac{2B}{\sigma_{oo}} \cos\theta \left(\sqrt{n^2-\sin^2\theta} - \cos\theta\right)\sqrt{1+R'^2+2R'\cos 2q_o d}\; e^{-i\psi_o} \quad (7)$$

with $$\psi_o = \tan^{-1} \frac{R' \sin 2q_o d}{1+R'\cos 2q_o d}. \quad (8)$$

§3.3. Calculation of Intensities of the First Order Lines.

In this section we shall calculate the intensity of the first order lines for normal incidence and for the angle of incidence for which the first order Bragg reflection (i.e. for θ such that $2\Lambda \sin\theta = \lambda$) occurs for cases where the perturbation calculation is valid. For this purpose, one has to solve for the amplitudes $N_{\ell m}$ in terms of the amplitude B of the incident wave, making use of §2.4 (10) to (12) and the various results derived in §3.1.

§3.3.1. Justification of the Neglect of Certain Reflection Terms.

One can simplify the solution of the equations (10) to (12) of §2.4 and the calculation of the intensities beyond the scattering medium on the basis of the following considerations: We have seen in §3.1 that, while the diagonal amplitudes $N_{\ell\ell}^{\pm}$ are determined by these equations, the non-diagonal amplitudes $N_{\ell+k,\ell}^{\pm}$ $(k \neq 0)$ are determined in terms of $N_{\ell\ell}^{\pm}$ by the recurrence relations §2.4 (9). The recurrence relations, which are the same for N^{+} and N^{-}, were solved in §3.1 by making use of a perturbation procedure. The main result of that section was, that for a given ℓ, the amplitudes fall off very rapidly as one moves away from the diagonal amplitudes $N_{\ell\ell}$. This implies that one has to consider only a few non-diagonal amplitudes, their number depending on the order of approximation to which one confines oneself. On making use of this fact, it follows from §2.4 (12) that

$$\left|\frac{N_{\ell\ell}^{-}}{N_{\ell\ell}^{+}}\right|^{2} \sim \left|\frac{-q_{\ell}+s_{\ell}}{q_{\ell}+s_{\ell}}\right|^{2} \tag{1}$$

For nearly normal incidence (θ in the ultrasonic diffraction experiments is at most 2^{0} or 3^{0})

$$\left|\frac{N^-_{\ell\ell}}{N^+_{\ell\ell}}\right|^2 \sim \left(\frac{n-1}{n+1}\right)^2. \qquad (2)$$

Since $(n - 1)^2/(n + 1)^2 \sim 0{\cdot}02$, one can neglect the amplitudes N^- altogether and determine the N^+'s from equations (10) and (11) of §2.4 alone.

This calculation of N^+ will give all the N^+'s real, while if the N^- terms were taken into account, they would be complex numbers of the kind given by equations (7) and (8) of §3.2. However, as q takes one of its numerous possible values, the phase angle ψ is, firstly, very small and, secondly, almost constant, e.g. to this crude approximation $|\psi| \leqslant \tan^{-1} R' \sim 0.02$ for every possible value of q. Hence in evaluating the N^+'s one could neglect entirely the effect of the N^-'s on them. It will be noticed that the recurrence relations contain only real parameters (Δ and f are real) and hence ψ for the non-diagonal amplitudes $N_{\ell+k,\ell}$ is the same as for $N_{\ell\ell}$.

The effect of N^- on the intensity of light of a given frequency beyond the scattering medium, however, will be negligible only if the summation over different m's (cf. equation (8) of §2.5) is effectively over a few m values. This is due to the fact that the N^-'s combine with different phases from the N^{+}'s in giving a particular B_{ℓ}. Now, as we shall see below, at least as long as the perturbation method is valid, only a few amplitudes $N_{\ell m}$ with given ℓ contribute to a given B_{ℓ} and the neglect of the contribution of the N^{-}'s is justified. In his work on ultrasonic diffraction, Brillouin (1933) assumed that the effect of reflected waves could be neglected. Since Brillouin's method is essentially valid under the same conditions as our perturbation method, his assumption is also justified. However, under experimental conditions for which the perturbation method is not valid, i.e. where a large number of N_{ℓ}^{+} and N_{ℓ}^{-} contribute to a given B_{ℓ}, it is by no means obvious that the effect of reflected waves can be neglected.

§3.3.2. Intensities of the First Order Lines for Normal Incidence.

Now we can proceed to solve the equations § 2.4 (10) and (11) for the $N_{\ell m}$ and thence calculate the intensity of the first order lines beyond the scattering medium. Since the perturbation treatment differs for the cases of normal and Bragg incidence, we first treat the case of normal incidence. In view of the discussion given in the previous section, N^- can be ignored altogether. For simplicity N^+'s will be written without the superscript + in the following. For normal incidence, one has to use the results of §3.1.3, Case IIb. For convenience, we first summarize the results obtained there. Since, now, $q_m^\circ = \pm\sqrt{\eta_m^\circ}$ we write the amplitude of the wave whose frequency is $\omega + \ell\Omega$ and whose q value is q_m°, $= N_\ell(q_m^\circ)$. Remembering further that $q_m = q_{-m}$ for normal incidence, (we now regard $m \geq 0$), it follows from §3.1.2 that to first order calculation

$$N_0(q_0) \neq 0, \quad N_{\pm 1}(q_0) = \frac{N_0(q_0)}{f_{\pm 1}(q_0)}\frac{\Delta}{2}, \quad N_{\pm k}(q_0) = 0 \text{ for } k \geq 2$$

$$N_1(q_1) \neq 0, \quad N_{-1}(q_1) \neq 0, \quad N_0(q_1) = \left[\frac{N_1(q_1)}{f_0(q_1)} + \frac{N_{-1}(q_1)}{f_0(q_1)}\right]\frac{\Delta}{2}, \tag{1}$$

and $N_{\pm 2}(q_1) = \dfrac{N_{\pm 1}(q_1)}{f_{\pm 2}(q_1)}\dfrac{\Delta}{2}$, while all other $N_{\pm k}(q_1) = 0$

Further $N_{\pm \ell}(q_\ell) \neq 0$, $N_{\pm\ell\pm k}(q_\ell) = \dfrac{N_{\pm\ell}(q_\ell)}{f_{\pm\ell\pm k}(q_\ell)}\dfrac{\Delta}{2}\delta_{k1}$.

Making use of the relations (1) and using the notation of § 3.1.1 equations (10) and (11) of § 2.5 reduce to

. . . .

$$0 = \frac{\sigma_k(q_k)\,N_k(q_k)}{q_k - s_k} + \frac{\Delta}{2}\left[\frac{\sigma_k(q_{k+1})\,N_{k+1}(q_{k+1})}{(q_{k+1}-s_k)\,f_k(q_{k+1})} + \frac{\sigma_k(q_{k-1})\,N_{k-1}(q_{k-1})}{(q_{k-1}-s_k)\,f_k(q_{k-1})}\right].$$

. . . .

$$0 = \frac{\sigma_1(q_1)\,N_1(q_1)}{q_1 - s_1} + \frac{\Delta}{2}\left[\frac{\sigma_1(q_2)\,N_2(q_2)}{(q_2 - s_1)\,f_1(q_2)} + \frac{\sigma_1(q_0)\,N_0(q_0)}{(q_0 - s_1)\,f_1(q_0)}\right]$$

$$\frac{2c^2 s_0 B}{\omega_0^2} = \frac{\sigma_0(q_0)\,N_0(q_0)}{q_0 - s_0} + \frac{\Delta}{2}\left[\frac{\sigma_0(q_1)\,N_1(q_1)}{(q_1 - s_0)\,f_0(q_1)} + \frac{\sigma_0(q_1)\,N_{-1}(q_1)}{(q_1 - s_0)\,f_0(q_1)}\right] \quad (2)$$

$$0 = \frac{\sigma_{-1}(q_1)\,N_{-1}(q_1)}{q_1 - s_{-1}} + \frac{\Delta}{2}\left[\frac{\sigma_1(q_0)\,N_0(q_0)}{(q_0 - s_{-1})\,f_{-1}(q_0)} + \frac{\sigma_{-1}(q_2)\,N_{-2}(q_2)}{(q_2 - s_{-1})\,f_{-1}(q_2)}\right]$$

. . . .

To a first approximation, the equations (2) give

$$\frac{\sigma_0(q_0)\,N_0(q_0)}{q_0 - s_0} = \frac{2c^2 s_0 B}{\omega_0^2} + O\left(\frac{\Delta}{2}\right)^2$$

$$\frac{\sigma_1(q_1)\,N_1(q_1)}{q_1 - s_1} = -\frac{\Delta}{2}\,\frac{2c^2 s_0 B}{\omega_0^2}\,\frac{\sigma_1(q_0)\,(q_0 - s_0)}{\sigma_0(q_0)\,(q_0 - s_1)\,f_1(q_0)} + O\left(\frac{\Delta}{2}\right)^2 \quad (3)$$

$$\frac{\sigma_{\pm k}(q_k)\,N_{\pm k}(q_k)}{q_k - s_k} = O\left(\frac{\Delta}{2}\right)^2 \text{ or higher for } k \geq 2.$$

Given the amplitudes $N_{\pm k}(q_k)$ from equations (3), we can calculate the amplitude B_ℓ of the first order line from § 2.5 (8) which may be written

$$\frac{2c^2 s_1 B_1}{\omega_1^2} = \frac{\sigma_1(q_1)\,N_1(q_1)}{q_1 - s_1}\, e^{i(q_1 - s_1)d}$$

$$+ \frac{\Delta}{2}\,\frac{\sigma_1(q_0)\,N_0(q_0)}{(q_0 - s_1)\,f_1(q_0)}\, e^{i(q_0 - s_1)d} + \cdots \quad (4)$$

Substituting from (3) in (4) and neglecting the small variations in σ, q, etc., in the amplitudes, we have

$$B_1 = \frac{\Delta}{2} \frac{(n^2-1)(n^2+2)}{3\alpha^2} B \left(e^{i(q_0-s_1)d} - e^{i(q_1-s_1)d} \right)$$

$$= \frac{\Delta}{2\alpha^2} B \left(e^{i(q_0-s_1)d} - e^{i(q_1-s_1)d} \right). \qquad (5)$$

since $\dfrac{(n^2-1)(n^2+2)}{3} \sim 1.$

Multiplying the right hand side of (5) by its complex conjugate, we have for the intensity of the first order line

$$I_1 = \frac{\Delta^2}{\alpha^4} B^2 \sin^2 \frac{(q_1-q_0)d}{2} \qquad (6)$$

It follows from equation (6) that the maximum intensity, $(I_1)_{max}$, of the first order line is given by

$$(I_1)_{max} = \frac{\Delta^2 B^2}{\alpha^4} \qquad (7)$$

We can use this result to fix an upper limit to the ultrasonic frequency beyond which no diffraction is likely to be observed for normal incidence. Assuming arbitrarily for this purpose that one is limited to measurements of 1% of the intensity of the incident light, we obtain for the maximum frequency Ω max beyond which no ultrasonic diffraction is likely to occur,

$$\Omega_{max} = \frac{v\sqrt{10\Delta}}{\lambda} \qquad (8)$$

where v is the velocity of sound in the medium. Now, for ultrasonic beams from quartz crystals, we have a maximum beam intensity of 10 watts/cm^2 and a corresponding maximum compression ratio of 10^{-4}. This gives Ω max $= 10^8$ cyc./sec. when v is assumed to have the value $v = 1.5 \times 10^5$ cm./sec. This upper limit is confirmed by the experiments of Bhagavantam and Rao (1948) who found that the ultrasonic diffraction was observable up to 0.5×10^8 to 1×10^8 cyc./sec.

It may be mentioned here that the lower limit of Ω for which ultrasonic diffraction occurs is determined mainly by various experimental limitations, such as aperture width, etc.

§3.3.3. Intensity of First Order Bragg Reflection.

If one neglects the small change in the wavelength of light due to the Doppler effect, the condition for Bragg reflection for an incident light of wavelength λ by an ultrasonic beam of wavelength Λ is

$$2\Lambda \sin\theta = j\lambda \tag{1}$$

where j is a positive integer when $\sin\theta$ is positive. For the first order Bragg reflection, $j = 1$. For this case (1) implies $2\sin\theta/\alpha = +1$. To determine the degeneracy in q^0 or η^0 values (cf. §3.1.3) we substitute this value of $2\sin\theta/\alpha$ in §3.1.3 (4) and obtain

$$l_2 - l_1 = 2l_2 + 1 \tag{2}$$

The case for which l_2 (or l_1) equals zero needs special treatment given under the heading Case 1 of §3.1.3 since, in this case, $|l_2 - l_1| = 1$. For any other integral value of l_2, $|l_2 - l_1| \geqslant 3$. Let the zeroth order eigenvalue of q corresponding to $l_2 = 0$ and $l_1 = -1$ be q^0. From (9) of §3.1.3, it follows that to a first approximation q^0 breaks up into two values

$$q_1 = \left[q^{0^2} + \frac{\Delta}{2}\sqrt{\frac{1}{f_0'(q^0)\,f_{-1}'(q^0)}}\right]^{1/2}. \tag{3a}$$

$$q_2 = \left[q^{0^2} - \frac{\Delta}{2}\sqrt{\frac{1}{f_0'(q^0)\,f_{-1}'(q^0)}}\right]^{1/2}. \tag{3b}$$

Writing for the amplitudes of the two waves, whose frequencies are ω ($l_2 = 0$) and whose q values are q_1 and q_2, the values $N_0(q_1)$ and $N_0(q_2)$, the corresponding amplitudes of the waves whose frequencies are $\omega - \Omega$ ($l_1 = -1$) are given by (cf. equation (10) of §3.1.3)

$$\left.\begin{aligned} N_{-1}(q_1) &= N_0(q_1)\sqrt{\frac{f_0'(q^0)}{f_{-1}'(q^0)}} \\ N_{-1}(q_2) &= -N_0(q_2)\sqrt{\frac{f_0'(q^0)}{f_{-1}'(q^0)}} \end{aligned}\right\} \quad (4)$$

Since, except for these amplitudes ($N_0(q_1)$, $N_0(q_2)$ $N_{-1}(q_1)$ and $N_{-1}(q_2)$), all the amplitudes are $\Delta/2$ times these or smaller (cf. §3.1.3 (11)), we neglect all the rest of the amplitudes in evaluating $N_0(q_1)$ and $N_0(q_2)$.

Remembering that N^-'s can be neglected as before, and writing for N^+ without the superscript +, the equations (10) and (11) of §2.4 reduce to

$$B = \frac{\omega_0^2}{2c^2 s_0}\left[\frac{\sigma_0(q_1)\,N_0(q_1)}{(q_1+s_0)} + \frac{\sigma_0(q_2)\,N_0(q_2)}{(q_2+s_2)}\right]$$

$$0 = \left[\frac{\sigma_{-1}(q_1)\,N_{-1}(q_1)}{(q_1+s_{-1})} + \frac{\sigma_{-1}(q_2)\,N_{-1}(q_2)}{(q_2+s_{-1}}\right]$$

where $s_l = +\sqrt{\frac{\omega_l^2}{c^2} - p_l^2}$. Writing $B' = B\,\frac{2c^2 s_0}{\omega_0^2}$, and making use of (4) these equations become

$$\left.\begin{aligned} B' &= \frac{\sigma_0(q_1)\, N_0(q_1)}{q_1+s_0} + \frac{\sigma_0(q_2)\, N_0(q_2)}{q_2+s_0} \\ \text{and} \quad 0 &= \frac{\sigma_{-1}(q_1)\, N_0(q_1)}{q_1+s_{-1}} + \frac{\sigma_{-1}(q_2)\, N_0(q_2)}{q_2+s_{-1}} \end{aligned}\right\} \tag{5}$$

which give

$$N_0(q_1) = \frac{B'}{A} \tag{6}$$

$$N_0(q_2) = \frac{B'}{A} \cdot \frac{q_2+s_{-1}}{q_1+s_{-1}} \cdot \frac{\sigma_{-1}(q_1)}{\sigma_{-1}(q_2)} \tag{7}$$

where

$$A = \frac{\sigma_0(q_1)}{q_1+s_0} + \frac{\sigma_0(q_2)}{q_2+s_0} \cdot \frac{q_2+s_{-1}}{q_1+s_{-1}} \cdot \frac{\sigma_{-1}(q_1)}{\sigma_{-1}(q_2)} \tag{8}$$

and

$$N_{-1}(q_1) = N_0(q_1) \times \nu \quad , \quad N_{-1}(q_2) = -N_0(q_2) \times \nu \tag{9}$$

with

$$\nu = + \sqrt{\frac{f_0'(q^0)}{f_{-1}'(q^0)}} \, .$$

Making use of (6) and (8) into §2.5 (8), we obtain for the amplitudes B_0 and B_{-1} of the transmitted and the Bragg reflected waves respectively as

$$B_0 = \frac{\omega_0^2}{2cs_0^2}\left[\frac{\sigma_0(q_1)\, N_0(q_1)\, e^{-iq_1 d}}{q_1+s_0} + \frac{\sigma_0(q_2)\, N_0(q_2)\, e^{-iq_2 d}}{q_2+s_0}\right] e^{-is_0 d} \tag{10}$$

$$B_{-1} = \frac{\omega_{-1}^2\, \nu}{2c^2 s_{-1}}\left[\frac{\sigma_{-1}(q_1)\, N_0(q_1)\, e^{-iq_1 d}}{q_1+s_{-1}} + \frac{\sigma_{-1}(q_2)\, N_0(q_2)\, e^{-iq_2 d}}{q_2+s_{-1}}\right] e^{-is_0 d} . \tag{11}$$

The expressions for the intensities I_0 and I_{-1} of these two waves, as calculated from (10) and (11) can be approximated, after some straight-forward algebra, by

$$I_0 \simeq B^2 \cos^2 \frac{(q_1 - q_2) d}{2} \tag{12a}$$

$$I_{-1} \simeq B^2 \sin^2 \frac{(q_1 - q_2) d}{2} \tag{12b}$$

where $q_1 - q_2$ can be easily evaluated from (3) and one obtains

$$q_1 - q_2 \simeq \frac{\Delta}{2} \frac{k}{n} . \tag{13}$$

It will be noticed that the sum $I_0 + I_{-1}$, is just B^2, i.e. the intensity of the incident wave. This is not surprising since in the above calculation of these intensities we had put the amplitudes of all the other waves inside the scattering medium equal to zero, in order to simplify the calculations.

In contrast to the case of normal incidence discussed in the previous section, the maximum intensity $(I_{-1})_{max}$ is the same as that of the incident light in this approximation. When first order terms, which we have neglected in this simple calculation, are taken into account, the intensity $(I_{-1})_{max}$ is slightly diminished but still remains of the same order of magnitude as the intensity of the incident light. So long as the perturbation procedure is valid (cf. §3.4), it is obvious, from §2.4 (10) and (11), that the intensities of all the other diffracted spectra will be of the order of $\frac{\Delta^2}{\alpha^4}$ B^2 or less. In particular, the maximum intensity $(I_{+1})_{max}$ of the first order line on the other side of the direct transmitted beam ($\ell = +1$) will be $\sim B^2 \Delta^2/\alpha^4$. As α increases, $(I_{+1})_{max}$ decreases to negligible values although $(I_{-1})_{max}$ remains of the order of B^2. This is in agreement with the experimental results of Bhagavantum and Rao (1948) where, while for low values of α both I_{+1} and I_{-1} occur in the diffracted spectra, as α increases I_{+1} diminishes in intensity and gradually disappears at about $\alpha = 0.015$. corresponding to a frequency of ultrasonic beam of 0.5×10^8 cyc./sec.

It will be noticed that the intensity of the first Bragg reflection varies periodically with the thickness of the scattering medium as was observed by Debye and Sears (1932).

§3.4. Criterion for the Validity of the Perturbation Method.

In this section, we derive the conditions under which the perturbation procedure to solve the recurrence relations §2.4 (9) is valid. These conditions, as will be seen below, give an indication of the number of orders likely to appear under given experimental conditions, i.e. for any given value of δ $(\equiv \Delta/\alpha^2)$ and the angle of incidence. Finally, we shall discuss how these results can be utilised to reduce the infinite set of equations, §2.4 (9) to (12), to a finite set of simultaneous equations of manageable proportions for any given value of δ.

§3.4.1. Deduction of the Criterion.

It is well known that, in order that the perturbation method be valid, the first non-zero corrections to any zeroth order characteristic value of the parameter (which has been denoted in our case by η) should be very much less than the difference between the two neighbouring zeroth order characteristic values. Now as we saw in §3.1, for all values of the integer ℓ except $\ell = 0, \pm 1$, and ± 2, the perturbation method of solving the recurrence relations §2.4 (9) can be carried out as though the characteristic values of the parameter were non-degenerate. We, therefore, derive the condition for the validity of the perturbation treatment for this case. Here, the first non-zero correction to is given by § 3.1.2 (3), (which gives $\eta_\ell^{(2)}$) times $(\Delta/2)^2$. Putting $\eta_\ell^{(2)} \cdot (\frac{\Delta}{2})^2$ equal to $C(\eta_\ell^0 - \eta_{\ell-1}^0)$ where C is some constant <u>sufficiently</u> less than unity and taking the absolute values, we have for the validity of the perturbation method, the condition

$$\left| \left(\frac{\Delta}{2}\right)^2 \frac{1}{f_\ell'(\eta_\ell^0)} \left[\frac{1}{f_{\ell+1}(\eta_\ell^0)} + \frac{1}{f_{\ell-1}(\eta_\ell^0)} \right] \right| \leq \left| C(\eta_\ell^0 - \eta_{\ell-1}^0) \right| \cdot \quad (1)$$

Making some simplifications and neglecting terms of the order $^v/c$ or smaller, we have from equation (1)

$$\left| \frac{(n^2-1)^2(n^2+2)^2}{18} \right| \leq \left| \frac{C\alpha}{\Delta^2} \left[2\sin\theta + \alpha(2\ell-1) \right]^2 \left[2\sin\theta + \alpha(2\ell+1) \right] \right| \quad (2)$$

where, it will be remembered,

This definition
was forgotten o
p. 36

$$\alpha = \frac{K}{k} = \frac{\lambda}{\Lambda} \quad (3)$$

For the sake of convenience in further discussion, we will take $\sin\theta$ to be positive; this, it will be seen, is not a restrictive assumption. Equation (2) can be immediately interpreted as giving two integers ℓ_1 and ℓ_2 beyond which the perturbation method is valid, namely for

$$\left.\begin{aligned} l \leqslant l_1 &\simeq -\frac{1}{2\alpha}\left[\frac{\Delta^2}{C\alpha}\,\frac{(n^2-1)^2(n^2+2)^2}{18}\right]^{1/3} - \frac{\sin\theta}{\alpha} \\ l \geqslant l_2 &\simeq \frac{1}{2\alpha}\left[\frac{\Delta^2}{C\alpha}\cdot\frac{(n^2-1)^2(n^2+2)^2}{18}\right]^{1/3} - \frac{\sin\theta}{\alpha} \end{aligned}\right\} \tag{4}$$

provided $\Delta \neq 0$. If $\Delta = 0$, it is immediately seen from (2) that l_1 and l_2 become indeterminate. $\Delta = 0$, however, corresponds to the homogeneous medium and there is then only the direct transmitted beam. Choosing, arbitrarily*, $C = 1/6$ and noting that $\frac{(n^2-1)(n^2+2)}{3} \sim 1$, the conditions (4) reduce to

$$l \leqslant l_1 \simeq -\left(\frac{\delta^2}{2}\right)^{1/3} - \frac{\sin\theta}{\alpha} \tag{5a}$$

$$l \geqslant l_2 \simeq \left(\frac{\delta^2}{2}\right)^{1/3} - \frac{\sin\theta}{\alpha} \tag{5b}$$

where δ is given by (cf. §1 (1))

$$\delta = \frac{\Delta}{\alpha^2}. \tag{6}$$

* We have chosen $C = 1/6$, for convenience. Since C occurs to one-third power in (4), a slightly different C will not affect the results appreciably. We should also mention here that the discussions of this section are essentially of a qualitative nature.

The conditions (5) obviously mean that the amplitudes $N_k(\eta_l)$ can be calculated in terms of $N_l(\eta_l)$ by applying the perturbation method to the recurrence relations §2.4 (19) so long as l does not lie between l_1 and l_2. Before we discuss the equations (5) and (6), we recall that if the perturbation method is valid for values of l near zero, the diagonal elements, N_{00}, $N_{\pm1\pm1}$, $N_{\pm2\pm2}$ .... as obtained by solving the equations (10) to (12) of §2.4, also form a very rapidly decreasing sequence. In view of this, it is obvious that, if the range of values of l for which the perturbation method is not valid does not include the $l = 0$ or ± 1, as is the case for very large values of $\sin\theta/\alpha$, such amplitudes are still negligible. For example, taking a hypothetical example, if the perturbation method is valid for all values of l except when l lies between -5 and -10, the only amplitudes of appreciable magnitude will be $N_k(\eta_l)$, (k and l both lying within the range -1 to 1); hence only the first order lines are likely to appear with slightly different intensities on either side of the direct transmitted beam. This situation can always occur for oblique incidence, howsoever large δ may be, for a large enough value of $\sin\theta$. In fact, by putting in (5b), $l_2 = -2$, say, we find a critical value of $\theta = \theta_c$ beyond which only first order lines, <u>if any</u>, will appear:

$$\sin\theta_c \simeq \theta_c \simeq 2\alpha + \left(\frac{\delta^2}{2}\right)^{1/3}\alpha = 2\alpha + \left(\frac{\Delta^2}{2\alpha}\right)^{1/3}. \qquad (7)$$

Taking a typical example, where $\Delta = 10^{-4}$, $\alpha = 0.003$ corresponding to an ultrasonic velocity of 1.5×10^5 cm./sec. and frequency of 10^7 cyc./sec. $\theta_c \sim 1^{\circ}.25'$. If one increases θ beyond this value, the intensity of the first order lines will decrease rapidly. This is in agreement with the experimental results that within the range of values of $\Delta \leqslant 10^{-4}$ and $\alpha = 3 \times 10^{-2}$ to 3×10^{-4}, no lines appear beyond θ equal to about 1.5° or so (see Figure 4).

For the case when the range of values of ℓ for which the perturbation method of solving the recurrence relations is not valid includes the value $\ell = 0$, it will be convenient to divide the discussion into two cases, viz., (a) normal incidence and (b) oblique incidence. We first discuss the case of normal incidence $(\theta = 0)$.

§3.4.2. <u>Case (a) Normal Incidence.</u>

Here the conditions (5) for the validity of the perturbation procedure reduce to

$$\left.\begin{aligned} l \leqslant l_1 &\simeq -\left(\frac{\delta^2}{2}\right)^{1/3} \\ l \geqslant l_2 &\simeq +\left(\frac{\delta^2}{2}\right)^{1/3} \end{aligned}\right\} \qquad (8)$$

The range of values for which the perturbation procedure is not valid depends on the value of the parameter δ. Now, for values of l where the perturbation procedure for obtaining the non-diagonal amplitudes $N_k(\eta_l)$ in terms of $N_l(\eta_l)$ is valid, it can be easily seen that equations §2.4 (10) to (12) imply the rapid falling off of the diagonal elements N_{ll} as one moves away from the range of values for which the perturbation procedure is not valid (cf. § 3.3.2). Hence the only amplitudes $N_k(\eta_l)$ which are more or less of the same order of magnitude are those for which both k and l lie between l_1 and l_2 and the amplitudes fall off very rapidly as one moves outside the range $l_1 \leqslant l \leqslant l_2$. It follows from (7), therefore, that numbers of orders L_1 and L_2 (with decreased and increased frequencies respectively) likely to appear are given by

$$L_1 = L_2 = \left(\frac{\delta^2}{2}\right)^{1/3} + 1 \text{ or } 2. \qquad (9)$$

It will be noticed that the numbers of orders likely to appear on either side of the direct transmitted beam are equal. This symmetry about the direct transmitted beam for normal incidence is fully confirmed experimentally.

For small values of δ ($\delta < 1$), it is seen from (9) that only first order lines will appear. For this case we have already given the expressions for the intensities of the first order lines (p. 50 of the thesis). For large values of δ, several lines will appear on either side of the direct transmitted beam. The relation (9) gives, then, the number of orders likely to appear under given experimental conditions, namely for $\theta = 0$ and for a given value of δ. In the plate of Figure 3 which was taken with $\theta = 0$, $\Delta \sim 10^{-4}$, $\alpha = 0.002$, $L_1 = L_2 \sim 9$ while the actual number of lines on the plate on either side ~ 15 which is in as good an agreement as could be expected in view of the uncertainty in the value of Δ and our arbitrary choice of C. The calculations of the intensities of the various orders for this case ($\delta \gg 1$), are complicated and it has not been possible for us to give explicit expressions. It may be pointed out that in these calculations for a given value of δ, out of the infinite set corresponding to l and m lying between $-L_1$ and $+L_2$ need be considered since all the other amplitudes

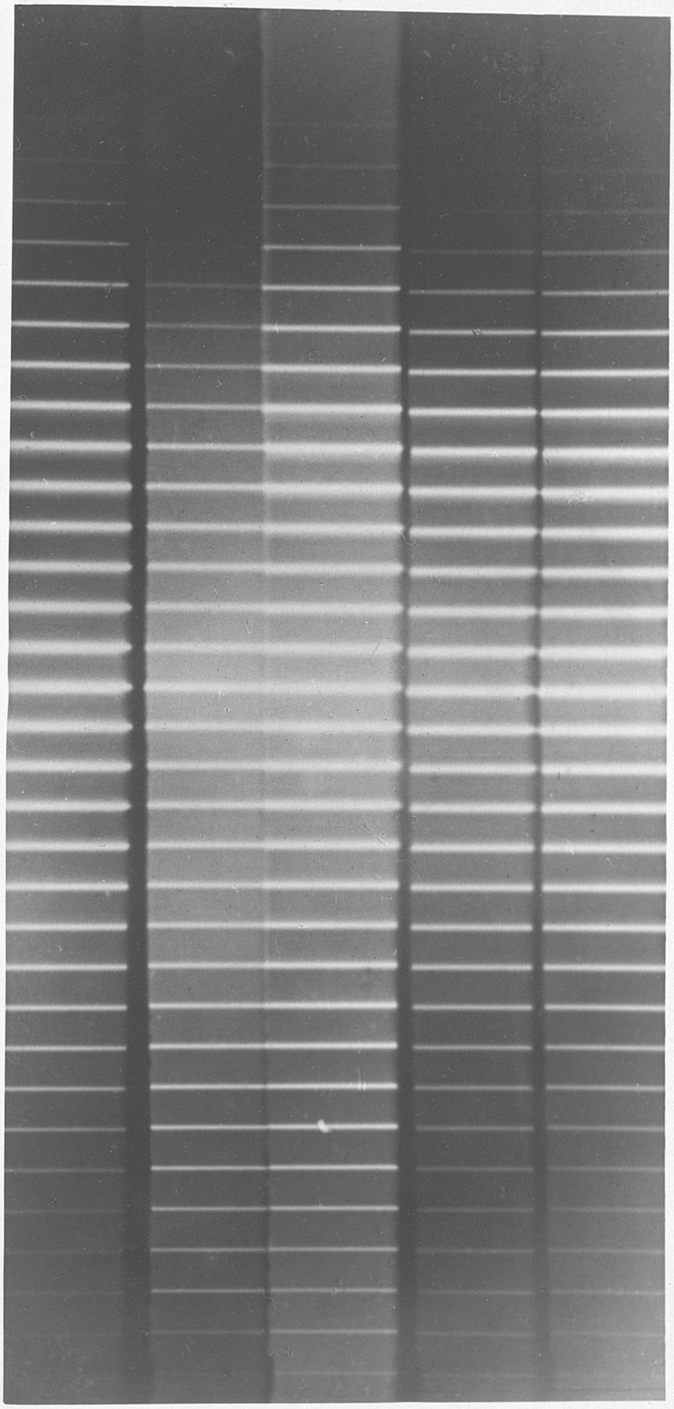

FIGURE 3.

Ultrasonic Diffraction for Normal Incidence.

(Original plate taken by George S. Stanford, in the Physics Laboratory, Acadia University) are in any case negligible. This, under extreme experimental conditions, involves solving at most 20 simultaneous equations and usually much less. Such a set of simultaneous equations with a finite number of unknowns can, in principle, always be solved.

We may mention here that $\delta \gg 1$ corresponds to the case discussed by Raman and Nath who obtained closed expressions for the intensities of the various orders. Their procedure, however, as mentioned in §1 appears to us lacking in mathematical justification. Nath (1936) in a later paper, in an attempt to remove this defect, obtained for the intensities of the various orders, infinite series which are only very slowly convergent <u>and sometimes even divergent</u> for some values of δ within the experimental range (cf. Bhagavantam and Rao (1948) p. 59). The present theory, we believe, is better in this respect since one can reduce the infinite set of equations (§2.4 (9) to (12) for the infinite number of unknowns to a finite set of equations with a finite number of unknowns. Moreover, it has been possible to obtain the number of orders likely to appear under given experimental conditions, i.e. for any given value of δ.

Before proceeding to the discussion of oblique incidence, it should be mentioned here that L_1 or L_2 should be regarded as the upper limit to the number of orders likely to appear under given experimental conditions, viz., for a given value of δ. This is due to the fact that although the various $N_k(\eta_\ell)$ (k and ℓ lying between L_1 and L_2) are not negligible, they may combine to give negligible intensity for a particular diffracted beam. For example, it may happen that even with many orders present the direct transmitted beam has zero intensity depending on the thickness of the scattering medium (the ultrasonic beam).

§3.4.3. Case (b) Oblique Incidence.

We have already discussed earlier in this section the case when $\sin\theta/\alpha$ is so large that the range of values of ℓ ($+\ell_1 \leqslant \ell \leqslant \ell_2$) for which the perturbation procedure is not valid does not include the value $\ell = 0$. There we deduced that for such values of θ only first order lines, if any, are likely to appear and their intensities will fall off as one increases θ from a certain critical value θ_c. When, however, $\theta < \theta_c$ and the range of values of for which the perturbation procedure is not valid includes the value $l = 0$, i.e. $\sin\theta/\alpha \leqslant \left(\frac{\delta^2}{2}\right)^{1/3}$, certain interesting features appear in the diffracted spectra. In this case, it follows from the arguments advanced in the preceding paragraphs that the number of orders L_1 and L_2 (with decreased and increased frequencies respectively) likely to appear for a given value of δ and $\sin\theta/\alpha$ are given by:

$$\left.\begin{aligned} L_1 &\simeq \frac{\sin\theta}{\alpha} + \left(\frac{\delta^2}{2}\right)^{1/3} + 1 \text{ or } 2 \\ L_2 &\simeq -\frac{\sin\theta}{\alpha} + \left(\frac{\delta^2}{2}\right)^{1/3} + 1 \text{ or } 2 \end{aligned}\right\} \tag{10}$$

where $\delta = \frac{\Delta}{\alpha^2}$ and $\left(\frac{\delta^2}{2}\right)^{1/3} > \frac{\sin\theta}{\alpha}$.

It will be noticed that for a given δ, as $\sin\theta/\alpha$ increases from zero the number of orders likely to appear on two sides of the direct transmitted beam become different. The number appearing with decreased frequencies gradually increases while the number with increased frequencies gradually diminishes. This qualitative feature deduced from the present theory is in contradiction of the theory of Raman and Nath, where in their approximation one obtains a complete symmetry about the direct transmitted beam in the diffracted spectra <u>even</u> for oblique incidence (cf. Willard (1949)). The results of the present theory are, however, in agreement with the experimental results. Parthasarathy (1936) has studied experimentally the diffracted spectra for different angles of incidence and we give in Figure 4 a plate from his paper. In this case $\alpha = 0.003$ and, assuming $\Delta \sim 10^{-4}$ as in §3.4.2, we have from (8) for normal incidence $L_1 = L_2 \sim 5$. The table below gives the number of lines actually appearing on either side of the direct transmitted beam at different angles of incidence; within brackets are the corresponding numbers likely to appear as given by (10) and the discussion on pages 60 and 61. From the table it is seen that the agreement is good for both the angles sufficiently smaller and the angles sufficiently larger than θ_c (cf. p. 61).

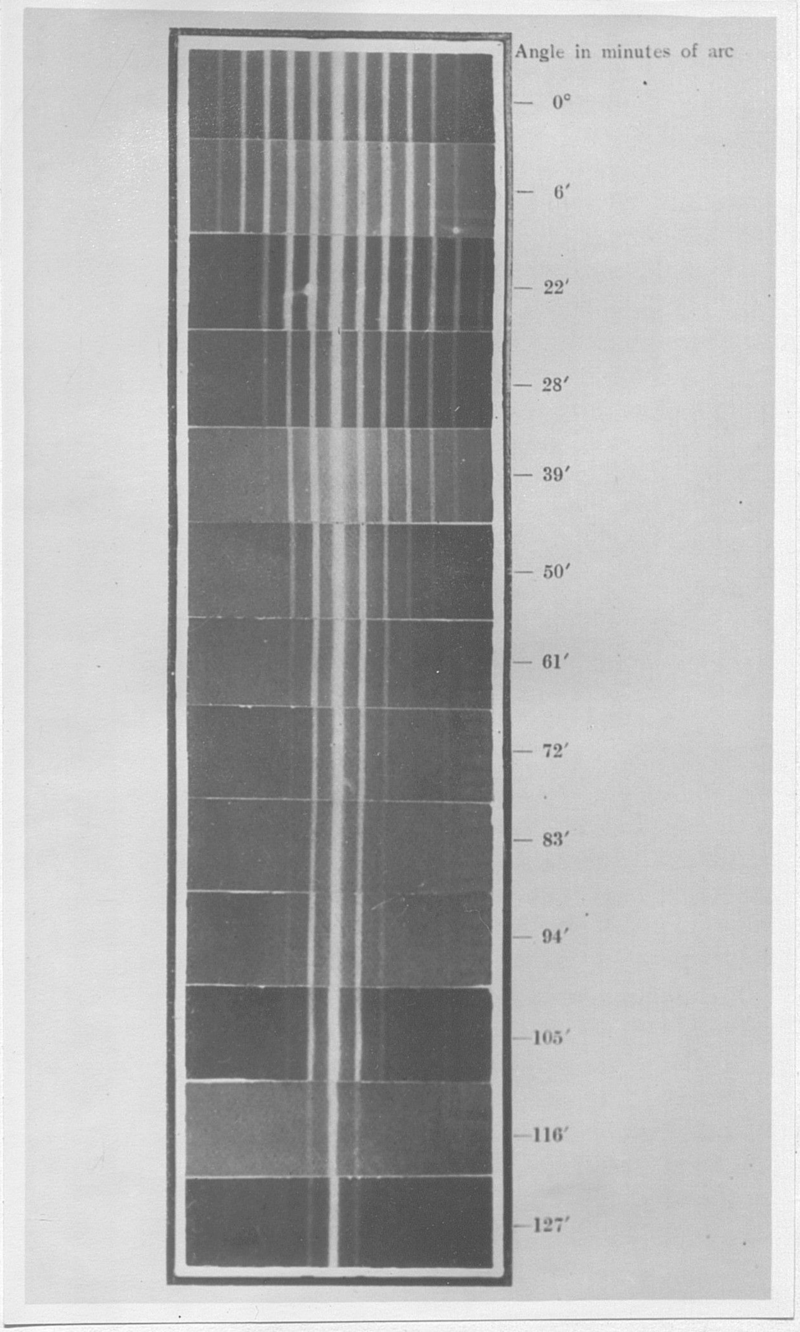

FIGURE 4.

Ultrasonic Diffraction for Oblique Incidence.

(Original plate by S. Parthasarathy in the Proc. Ind. Acad. Sci. A, 3, 442 (1936)).

TABLE OF NUMBERS OF SPECTRA.

Angle	$\sin\theta/\alpha$	Left Spectra	Right Spectra
0	0	5 (5)	5 (5)
6°	0.6	5 (4)	5 (6)
22°	2	3 (3)	6 (7)
39°	4	2 (1)	6 (9)
61°	6	2	3
83°	8	1	2
105°	10	1 (1 or 2)	1 (1 or 2)
127°	13	1 (1)	1 (1)

Finally, we mention that for oblique incidence, as for normal incidence, (for any given δ and $\sin\theta/\alpha$) the solving of various infinite sets of equations recurring in our theory can be reduced to the solving of a finite number of simultaneous equations with a finite number of unknowns.

IV. CONCLUSION AND SUMMARY.

In the first half of the thesis, we have rigorously formulated the problem of the diffraction of light by ultrasonic waves in terms of the scattering of electromagnetic waves by a periodically perturbed medium. This leads to an integral equation which has been solved formally with the help of a trial solution for the electric disturbance in the medium, in the form of a double infinity of plane waves. The condition that the trial function be a solution of the integral equation leads to (a) the frequencies and the directions of the diffracted spectra and (b) a set of equations for the amplitudes $N_{\ell m}$ of the components of the trial solution. The intensities of the diffracted spectra are immediately determined once the set of amplitudes $N_{\ell m}$ is known. The second half of the thesis is concerned with the approximate solutions of these equations and the deduction of various theoretical results which are then compared with the experimental facts.

It is shown that this theory is both valid and useful over the entire range of experimental conditions, i.e. for any given values of δ $(= \Delta/\alpha^2)$ and the angle of incidence θ. This is in contrast with the previous theories which are, at best, valid only over portions of the range. For small values of δ , only the first order lines appear and their intensities, both for normal and Bragg incidence, have been calculated. For large values of δ , the calculation of the intensities of the various orders is complicated but it is shown that such a calculation can always be performed. In this, unlike previous treatments, one is also able to deduce the number of orders likely to appear under given experimental conditions, i.e., for given values of δ and $\sin\theta$. These numbers as deduced from the theory are in satisfactory agreement with the experimental results of several workers. In particular, the well-known asymmetry in the diffracted spectra for oblique incidence receives from the present theory a natural explanation. The estimated asymmetry as a function of the angle of incidence is found to be in satisfactory agreement with the experimental results.

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